# Geometric Spanners with Applications in Wireless Networks *

Christian Schindelhauer ** Klaus Volbert** Martin Ziegler**

Institute of Computer Science
Heinz Nixdorf Institute
University of Paderborn, Germany
E-mail: {schindel,kvolbert,ziegler}@upb.de

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**Abstract**

In this paper we investigate the relations between spanners, weak spanners, and power spanners in $\mathbb{R}^{\mathcal{D}}$ for any dimension $\mathcal{D}$ and apply our results to topology control in wireless networks. For $c \in \mathbb{R}$, a *$c$-spanner* is a subgraph of the complete Euclidean graph satisfying the condition that between any two vertices there exists a path of length at most $c$-times their Euclidean distance. Based on this ability to approximate the complete Euclidean graph, sparse spanners have found many applications, e.g., in FPTAS, geometric searching, and radio networks. In a *weak* $c$-spanner, this path may be arbitrarily long, but must remain within a disk or sphere of radius $c$-times the Euclidean distance between the vertices. Finally in a *$c$-power* spanner, the total energy consumed on such a path, where the energy is given by the sum of the squares of the edge lengths on this path, must be at most $c$-times the square of the Euclidean distance of the direct edge or communication link.

While it is known that any $c$-spanner is also both a weak $C_1$-spanner and a $C_2$-power spanner for appropriate $C_1, C_2$ depending only on $c$ but not on the graph under consideration, we show that the converse is not true: there exists a family of $c_1$-power spanners that are not weak $C$-spanners and also a family of weak $c_2$-spanners that are not $C$-spanners for any fixed $C$. However a main result of this paper reveals that any weak $c$-spanner is also a $C$-power spanner for an appropriate constant $C$.

We further generalize the latter notion by considering $(c, \delta)$-power spanners where the sum of the $\delta$-th powers of the lengths has to be bounded; so $(c, 2)$-power spanners coincide with the usual power spanners and $(c, 1)$-power spanners are classical spanners. Interestingly, these $(c, \delta)$-power spanners form a strict hierarchy where the above results still hold for any $\delta \geq \mathcal{D}$; some even hold for $\delta > 1$ while counter-examples exist for $\delta < \mathcal{D}$. We show that every self-similar curve of fractal dimension $\mathcal{D}_f > \delta$ is not a $(C, \delta)$-power spanner for any fixed $C$, in general.

Finally, we consider the sparsified Yao-graph (SparsY-graph or YY) that is a well-known sparse topology for wireless networks. We prove that all SparsY-graphs are weak $c$-spanners for a constant $c$ and hence they allow us to approximate energy-optimal wireless networks by a constant factor.

## 1 Introduction

Geometric spanners were first introduced to computational geometry by Chew [3]. Peleg and Schaffer introduced them in the context of distributed computing [17]. They have applications in motion planning [4], they were used for approximating the minimum spanning tree [27], and for a fully polynomial time approximation scheme for the traveling salesman and related problems [2, 19]. A good survey of spanning trees and spanners is given by Eppstein in [7].

*A preliminary version of the results in this paper appeared in *Proc. of the 15th Annual International Symposium on Algorithms and Computation (ISAAC '04)* [21]. The applications extend some of the results presented in *Proc. of the 8th Annual European Conference on Parallel Processing (Euro-Par '02)* [10].

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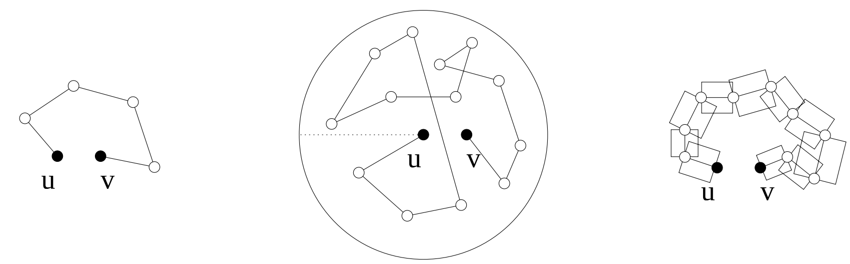
Roughly speaking, they approximate the complete Euclidean graph on a set of geometric vertices while having only linearly many edges. The formal condition for a $c$-spanner $G = (V, E)$ with $V \subset \mathbb{R}^{\mathcal{D}}$ is that between any two $u, v \in V$, the edge $(u, v)$ may be absent provided there exists a path in $G$ from $u$ to $v$ of length at most $c$-times the Euclidean distance between $u$ and $v$, see Figure 1(a). In particular, this path remains within a circle around $u$ of radius $c$. For applications in geometric searching [8, 9] and for optimizing routing time in wireless networks [10, 16], it has turned out that graphs with the latter, weaker condition suffice, see Figure 1(b). Several constructions yield both spanners [9] and weak spanners [8] with arbitrarily describable approximation ratio. Among them, some furthermore benefit from nice locality properties which led to successful applications in ad-hoc routing networks [10, 15, 25, 26].

However in order to restrict the power consumption during such a communication (which for physical reasons grows quadratically with the Euclidean length of each link), one is interested in routing paths, say between $u$ and $v$, whose sum of *squares* of lengths of the individual steps is bounded by $c$-times the square of the Euclidean distance between $u$ and $v$, see Figure 1(c). Such graphs are known as $c$-*power* spanners [10–15, 18]. Finally, when power consumption is of minor interest but the routing time




(a) Bounded length (b) Bounded radius (c) Bounded energy

Figure 1: Spanner, Weak Spanner and Power Spanner for $\mathcal{D} = 2$

is dominated by the number of individual steps, sparse graphs are desired which between any vertices $u$ and $v$ provides a path containing at most $c$ further vertices. These are the so-called $c$-*hop* spanners [1]. In this paper, we investigate the relations between these various types of spanners and apply our results to topology control in wireless networks.

Observe that any strongly connected finite geometric graph is a $C$-spanner for some value $C$. For this, e.g., consider for any pair $u, v$ of vertices some path from $u$ to $v$ and the ratio of its length to the distance between $u$ and $v$. Then taking for $C$ the maximum over the (finitely many) pairs $u, v$ will give the value $C$. Therefore the question on the relation between spanners and weak spanners rather asks whether any weak $c$-spanner is a $C$-spanner for some value $C$ depending *only* on $c$. Based on a construction from [6], we answer this to the negative. For some weak $c$-spanners it is proved that they are also $C$-power spanners for some value $C$ [10, 11] using involved constructions. One major contribution of our work generalizes and simplifies such results by showing that in $\mathbb{R}^{\mathcal{D}}$ in fact *any* weak $c$-spanner is a $C$-power spanner with $C = \mathcal{O}(c^{4\mathcal{D}})$. Moreover, we investigate the notion of a $(c, \delta)$-power spanner [10] which

- for $\delta = 1$ coincides with $c$-spanners
- for $\delta = 2$ coincides with (usual) $c$-power spanners
- for $\delta = 0$ coincides with $c$-hop spanners, i.e. graphs with diameter $c$
- for $\delta > 2$ reflects transmission properties of radio networks (e.g., for $\delta$ up to $6$ or even $8$ [20]) .

We show that these form a strict hierarchy: For $\Delta > \delta > 0$, any $(c, \delta)$-power spanner is also a $(C, \Delta)$-power spanner with $C$ depending only on $c$ and $\Delta/\delta$; whereas we give examples of $(C, \Delta)$-power spanners that are not $(c, \delta)$-power spanner for any fixed $c$. Our main contribution is that any weak $c$-spanner is also a $(C, \delta)$-power spanner for arbitrary $\delta \geq \mathcal{D}$ with $C$ depending on $c$, $\mathcal{D}$ and $\delta$ only. We finally show that this claim is best possible by presenting, for arbitrary $\delta < \mathcal{D}$, weak $c$-spanners which are not $(C, \delta)$-power spanner for any fixed $C$.

The remainder of this paper is organized as follows: In Section 2, we present preliminaries concerning geometric spanners and define the different types of spanners under consideration. In Section 3, we show that, while any $c$-spanner is also a weak $c$-spanner, a weak $c$-spanner is, in general, not a $C$-spanner for any $C$ depending just on $c$. Section 4 similarly reveals the relations between spanners and power spanners. In the main Section 5 of the this work, we investigate the relation between weak spanners and power spanners. Theorem 5.1 gives an example of a power spanner which is not a weak spanner. Our major contributions then prove that, surprisingly, any weak $c$-spanner is also a $C$-power spanner with $C$ depending only on $c$. For different values of $\delta$, we obtain different upper bounds to $C$ in terms of $c$: For $\delta = \mathcal{D}$, we show $C \leq \mathcal{O}(c^{4\mathcal{D}})$, see Theorem 5.6; for $\delta > \mathcal{D}$, we have $C \leq \mathcal{O}\left(c^{\mathcal{D}+\delta}/(1-2^{\mathcal{D}-\delta})\right)$, see Theorem 5.5. However for $\delta < \mathcal{D}$, we present counter-examples of unbounded $C$, that is, in this case provably not any weak $c$-spanner is a $(C,\delta)$-power spanner. Furthermore, we generalize our construction and analysis to self-similar fractal curves. Section 6 finally shows that for different $\delta$, the respective classes of $(c,\delta)$-power spanners form a strict hierarchy. In Section 7, we present applications of our results concerning power-efficient wireless networks, before we conclude this work in Section 8.

# 2 Geometric Spanners

We model wireless networks using geometric graphs. A geometric graph $G = (V, E)$ consists of a set of vertices (or nodes) $V \subset \mathbb{R}^{\mathcal{D}}$ for $\mathcal{D} \in \mathbb{N}$ and a set of edges (or links) $E$. We define the size of $G$ as the number of nodes contained in $G$ denoted by $|V|$. For all $u, v \in V$, let $(u, v)$ denote a directed edge from $u$ to $v$, and $\{u, v\}$ denote an undirected edge connecting $u$ and $v$. We call $G$ undirected if $E \subseteq \{\{u, v\} \mid u, v \in V\}$, and directed if $E \subseteq \{(u, v) \mid u, v \in V\}$. For all $u, v \in V$ let $|u - v|$ be defined as the Euclidean distance between $u$ and $v$, this is, for completeness, $\sqrt{\sum_{i=1}^{\mathcal{D}}(u_i - v_i)^2}$. A finite sequence $P = (u = u_1, u_2, \ldots, u_\ell = v)$ of nodes $u_i \in V$ such that $(u_{i-1}, u_i) \in E$ for all $i \in \{2, \ldots, \ell\}$ is called a path from $u$ to $v$ in $G$. Occasionally, we also encounter the more general situation of a path from $u$ to $v$ that is not necessarily in $G$. This means that $u_i \in V$ still holds, but the requirement $(u_{i-1}, u_i) \in E$ is dropped. The radius of $P$ is the real number $\max_{i=1,\ldots,\ell} |u - u_i|$. The (Euclidean) length of P is given by $\sum_{i=2}^{\ell} |u_i - u_{i-1}|$. Then the hop length is $\ell - 1$ and for $\delta \geq 0$ we define the $\delta$-cost of a path $P$ by

$$||P||^\delta := \sum_{i=2}^{\ell} |u_i - u_{i-1}|^\delta \ .$$

The length is just the 1-cost whereas the hop length coincides with the 0-cost. The $\delta$-cost for $\delta \geq 2$ reflects the transmission properties of radio networks. In this case $\delta$ is also called the propagation exponent [20]. The graph $G$ is called connected if for every pair of nodes $u, v \in V$, there is a path in $G$ from $u$ to $v$. If $\{u, v\} \in E$ then $u$ is called a neighbor of $v$. The number of neighbors of $v$ gives the degree of $v$ denoted by deg$(v)$. The degree of $G$ is defined as deg$(G) := \max_{v \in V}$ deg$(v)$. For directed graphs, we define the number of edges ending at a node $v$ the in-degree of $v$, and the number of edges leaving $v$ the out-degree of $v$.

**Definition 2.1** *Let $G = (V, E)$ be a geometric directed graph with finite $V \subset \mathbb{R}^{\mathcal{D}}$ and $c > 0$. $G$ is a* $c$-spanner, *if for all $u, v \in V$ there is a path $P$ from $u$ to $v$ in $G$ of length $\|P\|^1$ at most $c \cdot |u - v|$. $G$ is a* weak $c$-spanner, *if for all $u, v \in V$ there is a path $P$ from $u$ to $v$ in $G$ of radius at most $c \cdot |u - v|$. For $\delta \geq 0$, $G$ is a $(c, \delta)$-*power spanner *if for all $u, v \in V$ there is a path $P$ from $u$ to $v$ in $G$ of $\delta$-cost $\|P\|^\delta$ at most $c \cdot |u - v|^\delta$. $G$ is a $c$-*power spanner, *if $G$ is a $(c, 2)$-power spanner. The factor $c$ is called* length stretch factor, weak stretch factor *or* power stretch factor, *respectively.*

Informally (see Figure 1), in a $c$-spanner there exists between two arbitrary vertices a path of length at most $c$-times the Euclidean distance between these vertices (bounded length). In a weak $c$-spanner, this path may be arbitrary long but must remain within a disk or sphere of radius $c$-times the Euclidean distance between the vertices (bounded radius). Finally in a $c$-power spanner, the energy consumed on such a path (e.g., the sum of the *squares* of the lengths of its constituting edges) must be at most $c$-times the one consumed on a putative direct link (bounded cost).

Later on, we shorten the notion of spanner, weak spanner and power spanner and omit constant parameters. Furthermore, we concentrate on families of graphs. A family of graphs

$$\mathcal{G} = \{G_n \mid G_n \text{ is a geometric graph for } n \in \mathbb{N}\}$$

fulfills a given property if and only if $G_n$ fulfills this property for all $n \in \mathbb{N}$. If it is clear to which family a graph belongs, we say that this graph fulfills a property if and only if its family fulfills the same property. For example, if we say that a family of graphs is a *spanner*, then there exists a constant $c$ such that all its members are $c$-spanners.

The attentive reader might have observed that our Definition 2.1 does not exactly match that from [10]. The latter required that the $\delta$-cost of some path $P$ from $u$ to $v$ in $G$ is bounded by $c$-times the $\delta$-cost of *any* path $Q$ (not necessarily in $G$) from $u$ to $v$. However, both approaches are in fact equivalent: let $G = (V, E)$ be a $(c, \delta)$-power spanner, $u, v \in V$, and let $Q$ denote some path $Q = (u = u_1, \ldots, u_\ell = v)$ (not necessarily in $G$) from $u$ to $v$ of minimum $\delta$-cost. For each $i = 2, \ldots, \ell$ there exists by presumption a path $P_i$ in $G$ from $u_{i-1}$ to $u_i$ of $\delta$-cost at most $c \cdot |u_i - u_{i-1}|^\delta$. The concatenation of all these paths yields a path $P$ from $u$ to $v$ in $G$ with $\delta$-cost $\|P\|^\delta$ at most $c \cdot \|Q\|^\delta$.

# 3 Spanners versus Weak Spanners

Every $c$-spanner is also a weak $c$-spanner. Our first result shows that the converse is not true, in general.

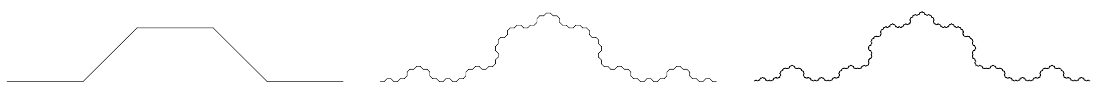


Figure 2: EPPSTEIN'S construction: a fractal curve with high dilation (not a spanner but a weak spanner)

**Theorem 3.1** *There is a family of graphs $\mathcal{G} = (V, E)$ with $V \subset \mathbb{R}^\mathcal{D}$ all of which are weak $(\sqrt{3} + 1/2)$-spanners but not $C$-spanners for any fixed $C \in \mathbb{R}$.*

**Proof:** We show the claim using the fractal construction presented in [6] (see Figure 2). We briefly review its recursive definition which is similar to that of a KOCH Curve. At the beginning, there are two vertices with distance 1. In the following steps we replace each edge by 5 new edges of equal length as follows: one horizontal, one at angle $\pi/4$, a second horizontal, another one at angle $-\pi/4$ and a third horizontal. After $i$ steps we have a graph consisting of $5^i$ edges and $5^i + 1$ vertices. As shown in [6] this graph has unbounded length stretch factor. We argue that there exists a constant $c$ such that it is a weak $c$-spanner. It is known that the area under the constructed curve is bounded by a constant and that the path between two vertices $u, v \in V$ lies completely in a disk around the midpoint of the segment between $u$ and $v$ with radius at most $(2 \cdot \sqrt{3}/2) = \sqrt{3}$ (see KOCH's Snowflake, Figure 10). Applying Observation 1 proves the claim. ■

The following observation says that, except for constants, it makes no difference in the definition of a weak spanner whether the radius is bounded with respect to center $u$ (the starting one of the two points) or with respect to center $(u + v)/2$ (the midpoint of the segment between the two points).

**Observation 1** *Let $P = (u = u_1, \ldots, u_\ell = v)$ be a path in the geometric graph $G = (V, E)$ such that $|u - u_i| \leq c \cdot |u - v|$ for all $i = 1, \ldots, \ell$. Then $w := (u + v)/2$ satisfies by the triangle inequality*

$$|w - u_i| \quad = \quad |u - u_i + (v - u)/2| \quad \leq \quad |u - u_i| + |v - u|/2 \quad \leq \quad (c + \tfrac{1}{2}) \cdot |u - v| \ .$$

*Conversely if $P$ has $|w - u_i| \leq c \cdot |u - v|$ for all $i$, then*

$$|u - u_i| \quad = \quad |w - u_i + (u - v)/2| \quad \leq \quad |w - u_i| + |u - v|/2 \quad \leq \quad (c + \tfrac{1}{2}) \cdot |u - v| \ .$$

# 4 Spanners versus Power Spanners

The first result of this section shows that, for $\delta > 1$, every $c$-spanner is also a $(c^\delta, \delta)$-power spanner.

**Theorem 4.1** *For $\delta > 1$, every $c$-spanner is also a $(c^\delta, \delta)$-power spanner.*

**Proof:** Let $G = (V, E)$ be a $c$-spanner, $u, v \in V$, and $P_{\mathtt{OPT}} = (u = u_1, u_2, \ldots, u_\ell = v)$ be an optimal path from $u$ to $v$ concerning the $\delta$-cost (not necessarily in $G$). Since $G$ is a $c$-spanner, for each edge $(u_i, u_{i+1})$ on the path $P_{\mathtt{OPT}}$, there is a path $P_i = (u_i = w_1, w_2, \ldots, w_{\ell_i} = u_{i+1})$ in $G$ with $||P_i|| = \sum_{j=1}^{\ell_i - 1} |w_j - w_{j+1}| \leq c \cdot |u_i - u_{i+1}|$. Now let $P$ be the concatenation of all these paths $P_i$ for $i = 1, \ldots, \ell - 1$, then we get a path $P$ from $u$ to $v$ in $G$ with:

$$||P||^\delta = \sum_{i=1}^{\ell-1} ||P_i||^\delta \leq \sum_{i=1}^{\ell-1} (c \cdot |u_i - u_{i+1}|)^\delta = c^\delta \cdot \sum_{i=1}^{\ell-1} (|u_i - u_{i+1}|)^\delta = c^\delta \cdot ||P_{\mathtt{OPT}}||^\delta$$

■

However, conversely, for any $\delta > 1$, there are $(c, \delta)$-power spanners which are not $C$-spanners for any fixed $C$: this follows from Theorem 5.1 presented below, in Section 5, as any $C$-spanner is a weak $C$-spanner as well.

# 5 Weak Spanners versus Power Spanners

Now we turn to the main contribution of this work and present our results concerning the relation between weak spanners and power spanners. Surprisingly, it turns out that any weak $c$-spanner is also a $C$-power spanner for some $C$ depending only on $c$. But first observe that the converse is not true, in general:

**Theorem 5.1** *In the plane (also in $\mathbb{R}^{\mathcal{D}}$) and for any $\delta > 1$, there is a family of $(c, \delta)$-power spanners which are not weak $C$-spanners for any fixed $C$.*

**Proof:** Let $V := \{u = v_1, \ldots, v_n = v\}$ be a set of $n$ vertices placed on a circle scaled such that the Euclidean distance between $u$ and $v$ is 1 and $|v_i - v_{i+1}| = 1/i$ for all $i = 1, \ldots, n-1$. Now consider the graph $G = (V, E)$ with edges $(v_i, v_{i+1})$. First observe that $G$ is a $(c, \delta)$-power spanner with $c$ independent of $n$. Indeed, its $\delta$-power stretch factor is dominated by the $\delta$-cost of the (unique) path $P$ in $G$ from $u$ to $v$ which amounts to

$$\|P\|^\delta \quad = \quad \sum_{i=1}^{n-1} (1/i)^\delta \quad \leq \quad \sum_{i=1}^{\infty} (1/i)^\delta \quad =: \quad c$$

a convergent series since $\delta > 1$. This is compared to the cost of the direct link from $u$ to $v$ of 1. On the other hand, the Euclidean length (the 1-cost) of the polygonal chain from $u$ to $v$ is given by the unbounded harmonic series $\sum_{i=1}^{n-1} (1/i) = \Theta(\log n)$. Therefore the radius of this polygonal chain also cannot be bounded by any $C$ independent of $n$, either. ■

Subsequently, we show that, conversely, any weak $c$-spanner is a $(C, \delta)$-power spanner for both $\delta > \mathcal{D}$ (Subsection 5.1) and $\delta = \mathcal{D}$ (Subsection 5.2) with $C$ depending only on $c$, $\mathcal{D}$ and $\delta$. A counter-example in Subsection 5.3 reveals that this, however, does not hold for $\delta < \mathcal{D}$.

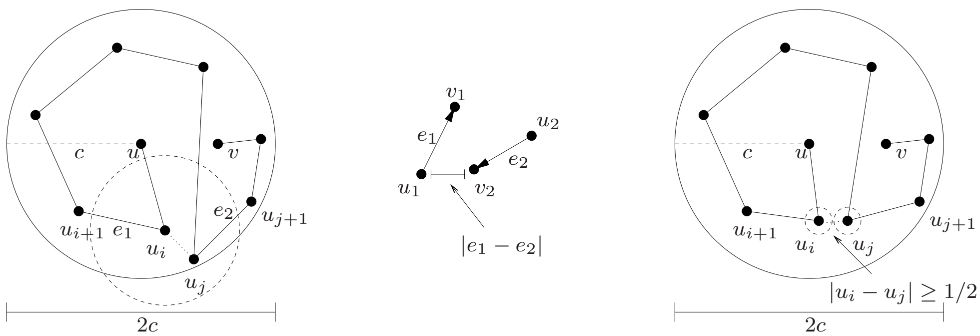
## 5.1 Weak Spanners are Power Spanners for $\delta > \mathcal{D}$

In this subsection, we show that any weak $c$-spanner is also a $(C, \delta)$-power spanner for any $\delta > \mathcal{D}$ with $C$ depending only on $c$, $\mathcal{D}$ and $\delta$. By definition between vertices $u, v$, there exists a path $P$ in $G$ from $u$ to $v$ that remains within a disk or sphere around $u$ of radius $c \cdot |u - v|$. However on the course of this path, two of its vertices $u'$ and $v'$ might come very close so that $P$, considered as a subgraph of $G$, in general, is not a weak $c$-spanner. On the other hand, $G$ being a weak $c$-spanner, there also exists a path $P'$ of small radius between $u'$ and $v'$. Based on such repeated applications of the weak spanner property, we first assert the existence of a path which, considered as a subgraph of $G$, is a weak $2c$-spanner.

**Definition 5.2** *Let $G = (V, E)$ be a directed geometric graph and $e_1 := (u_1, v_1)$, $e_2 := (u_2, v_2)$ two of its edges. By their* distance *we mean the number*

$$\min\left\{|u_1 - u_2|, |v_1 - v_2|, |u_1 - v_2|, |v_1 - u_2|\right\} \ ,$$

*that is, the Euclidean distance of the closest pair of their vertices (see Figure 3(b)).*



(a) Improving paths (b) Edge distance (c) Counting edges

Figure 3: Construction of a path with low power stretch factor in a weak spanner for $\mathcal{D} = 2$

**Lemma 5.3** *Let $G = (V, E)$ be a weak $c$-spanner and $u, v \in V$. Then there is a path $P$ from $u$ to $v$ in $G$ which, as a subgraph of $G$, is a weak $2c$-spanner.*

**Proof:** We consider a path $P$ from $u$ to $v$ in $G$ that fulfills the weak spanner property and modify this path step by step until the required property is guaranteed. The idea is to locally replace each part of $P$ connecting vertices $u'$ and $v'$ that violates the weak spanner property in $G(P)$ by a path from $u'$ to $v'$ in $G$. However for these iterated improvements to eventually terminate, we perform them in decreasing order of the lengths of the edges involved.

W.l.o.g. we assume $|u-v| = 1$. Since $G$ is a weak $c$-spanner, there exists a path $P = (u = u_1, \ldots, u_\ell = v)$ from $u$ to $v$ in $G$ that lies completely within a disk or sphere around $u$ of radius $c$. In particular, any edge on this path has a length of at most $2c$, see Figure 3(a).

Now consider all edges on this path of length between $c$ and $2c$. For any pair $e_1 = (u_i, u_{i+1})$ and $e_2 = (u_j, u_{j+1})$ with $j > i$ closer than $\frac{1}{2}$ (Definition 5.2), w.l.o.g. let $u_i$ and $u_j$ be the closest pair of their vertices, replace the path from $u_i$ to $u_j$ with a path according to the weak spanner property. This improvement is applied to vertices of distance at most $\frac{1}{2}$, so this sub-path remains within a disk or sphere of radius $c/2$; in particular, any edge introduced to $P$ has length at most $c$ and thus does not affect the edges of length between $c$ and $2c$ currently considered. Moreover, after having performed such improvements to all edges of length between $c$ and $2c$, the modified path $P$ has radius $c + c/2$, although it might now leave the disk around $u$ of radius $c$.

Next, we apply the same process to edges of length between $c$ and $c/2$ and perform improvements on those closer than $\frac{1}{4}$. The path $P$ thus obtained remains within a disk or sphere of radius $c + c/2 + c/4$ while, for any pair of vertices $u'$ and $v'$ improved in the previous phase, the sub-path between them might increase in radius from $c \cdot |u' - v'|$ to at most $(c + c/2) \cdot |u' - v'|$.

As $G$ is a finite graph, repeating this process for edges of length between $c/2$ and $c/4$ and so on, will eventually terminate and yield a path $P$ from $u$ to $v$ remaining within a disk or sphere of radius $c + c/2 + c/4 + \ldots = 2c$. Moreover, for any pair of vertices $u', v'$ in $P$, the sub-path between them has radius at most $(c + c/2 + c/4 + \ldots) \cdot |u' - v'|$ which proves that $P$ is indeed a weak $2c$-spanner. ■

**Lemma 5.4** *Let $P = (u_1, \ldots, u_\ell)$ be a weak $2c$-spanner, $u_i \in \mathbb{R}^{\mathcal{D}}$, $|u_1 - u_\ell| = 1$. Then $P$ contains at most $(8c+1)^{\mathcal{D}}$ edges of length greater than $c$; more generally, $P$ contains at most $(8c+1)^{\mathcal{D}} \cdot (2^{\mathcal{D}})^k$ edges of length greater than $c/2^k$.*

**Proof:** Let $k_{\mathcal{D}} := \frac{\pi^{\mathcal{D}/2}}{(\mathcal{D}/2)!}$, then $k_{\mathcal{D}} r^{\mathcal{D}}$ is the volume of a $\mathcal{D}$-dimensional sphere of radius $r$. Consider two edges $(u_i, u_{i+1})$ and $(u_j, u_{j+1})$ on $P$ both of length at least $c$ with $j > i$. $P$ being a weak $2c$-spanner implies that, between vertices $u_i$ and $u_j$, the sub-path in $P$ from $u_i$ to $u_j$ (which is unique and passes through $u_{i+1}$), satisfies $c \leq |u_i - u_{i+1}| \leq 2c \cdot |u_i - u_j|$; hence, $|u_i - u_j| \geq \frac{1}{2}$, see Figure 3(c). In particular, placing an Euclidean disk or sphere $B_i$ of radius $\frac{1}{4}$ around each starting vertex $u_i$ of an edge of length at least $c$ results in these disks being mutually disjoint. If $m$ denotes the number of edges of length at least $c$, these disks thus cover a total area of $mk_{\mathcal{D}}(\frac{1}{4})^{\mathcal{D}}$. On the other hand, as all $u_i$ lie within a single disk or sphere around $u_1$ of radius $2c$, all disks $B_i$ together cover an area of at most $k_{\mathcal{D}}(2c + \frac{1}{4})^{\mathcal{D}}$. Therefore,

$$m \quad \leq \quad \frac{k_{\mathcal{D}}(2c + \frac{1}{4})^{\mathcal{D}}}{k_{\mathcal{D}}(\frac{1}{4})^{\mathcal{D}}} \quad = \quad (8c+1)^{\mathcal{D}} \ .$$

For edges $(u_i, u_{i+1})$ and $(u_j, u_{j+1})$ on $P$ longer than $c/2^k$, one similarly obtains $|u_i - u_j| \geq 2^{-k-1}$ so that, here, Euclidean disks or spheres of radius $2^{-k-2}$ can be placed mutually disjoint within the total area of $k_{\mathcal{D}}(2c + 2^{-k-2})^{\mathcal{D}}$. ■

**Theorem 5.5** *Let $G = (V, E)$ be a weak $c$-spanner with $V \subset \mathbb{R}^{\mathcal{D}}$. Then $G$ is a $(C, \delta)$-power spanner for $\delta > \mathcal{D}$ where $C := (8c+1)^{\mathcal{D}} \cdot \dfrac{(2c)^{\delta}}{1 - 2^{\mathcal{D}-\delta}}$.*

**Proof:** Fix $u, v \in V$, w.l.o.g. $|u - v| = 1$. In the following we analyze the $\delta$-cost of the path $P$ constructed in Lemma 5.3 for $\delta = \mathcal{D} + \epsilon$. We consider all edges on this path and divide them into classes depending on their lengths. According to Lemma 5.4, there are at most $(8c+1)^{\mathcal{D}}$ edges of length between $c$ and $2c$, each one inducing $\delta$-cost at most $(2c)^{\delta}$. More generally, we have at most $(8c+1)^{\mathcal{D}} \cdot (2^{\mathcal{D}})^k$ edges of length between $c/2^k$ and $2c/2^k$ and the $\delta$-cost of any such edge is at most $(2c/2^k)^{\delta}$. Summing up over all possible edges of $P$, thus yields a total $\delta$-cost of $P$ of at most

$$\|P\|^{\delta} \quad \leq \quad \sum_{k=0}^{\infty} (8c+1)^{\mathcal{D}} \cdot (2^{\mathcal{D}})^k \cdot \left(\frac{2c}{2^k}\right)^{\delta} \quad = \quad (8c+1)^{\mathcal{D}} \cdot \frac{(2c)^{\delta}}{1 - 2^{\mathcal{D}-\delta}}$$

■

## 5.2 Weak Spanners are Power Spanners for $\delta = \mathcal{D}$

The preceding subsection showed that, for fixed $\delta > \mathcal{D}$, any weak $c$-spanner is also a $(C, \delta)$-power spanner. The present subsection yields the same for $\delta = \mathcal{D}$, a case which, however, turns out to be much more involved. Moreover, our bounds on $C$ in terms of $c$ and $\mathcal{D}$ become slightly worse. In fact, the most significant result of this work is the following:

**Theorem 5.6** *Let $G = (V, E)$ be a weak $c$-spanner with $V \subset \mathbb{R}^{\mathcal{D}}$. Then $G$ is a $(C, \mathcal{D})$-power spanner for $C := \mathcal{O}(c^{4\mathcal{D}})$.*

**Proof:** First recall that between vertices $u, v \in V$ there is a path $P$ in $G$ from $u$ to $v$ which remains inside a square or ($\mathcal{D}$-dimensional) cube of length $\ell := 2c \cdot |u - v|$ and center $u$. We denote such a square or cube by $S_u(\ell)$. By $s$ we denote the starting point of the path and by $t$ the end (target) point. We denote by $V(P)$ the vertex set of a path and by $E(P)$ the edge set of a path.

We give a constructive proof of the Theorem, i.e. given a path in $G$ obeying the weak spanner property we construct a path which obeys the $\big(\mathcal{O}(c^{4\mathcal{D}}), \mathcal{D}\big)$-power spanner property. For this we iteratively apply a procedure called **clean-up** to a path, yielding paths with smaller and smaller costs.

Besides the path $P$ in $G$ this procedure has parameters $L, d, D \in \mathbb{R}^+$. Hereby, $L$ denotes the edge length of a square or cube with central point $s$ containing the whole path. The parameters $d, D$ are in the range $0 < 3(2c\sqrt{\mathcal{D}+3} + 2)d \leq D \leq L$ and can be chosen arbitrarily, yet fulfilling $D/d \in \mathbb{N}$ and $L/D \in \mathbb{N}$. These parameters define two edge-parallel ($\mathcal{D}$-dimensional) grids $G_d$ and $G_D$ of grid size $d$ and $D$ such that boundaries of $G_D$ are also edges of $G_d$. These grids fill out the square or cube $S_u(L)$, while the boundary edge of $S_u(L)$ coincides with the boundary of $G_d$ and $G_D$, see Figure 4. The outcome of the

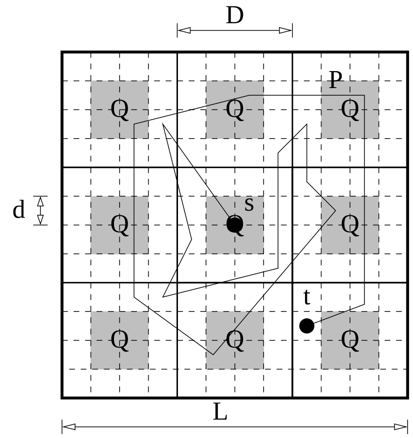



Figure 4: Idea and most important parameters for the proof of Theorem 5.6 for $\mathcal{D} = 2$

procedure **clean-up** is a path $P' = \text{clean-up}(P, L, d, D)$ which reduces the cost of the path while obeying other constraints, as we show shortly.

In Figure 6 we describe the procedure **clean-up** which uses the procedure **contract** described in Figure 5. Let $D(A)$ denote the diameter of the area or volume $A$.

**Lemma 5.7** *Let* $P = (v_1, \ldots, v_m)$ *and* $P' = \textbf{contract}(P, A) = (v_1, \ldots, v_{i-1}, v_i = w_1, \ldots, w_k = v_j, v_{j+1}, \ldots, v_m)$*. Then the following properties are satisfied.*

- *Locality:* $\forall u \in \{w_1, \ldots, w_k\}$*:* $\min_{p \in A} |u - p| \leq c \cdot D(A)$ *and* $\max_{p \in A} |u - p| \leq (c + 1) \cdot D(A)$*.*
- *Continuity of long edges:* $\forall e \in E(P') : |e| > 2c \cdot D(A) \Longrightarrow e \in E(P)$*.*

**Proof:** The maximum distance between $v_i$ and $v_j$ is at most $D(A)$. The replacement path $(w_1, \ldots, w_k)$ is inside a disk or sphere of radius $c \cdot D(A)$. Hence for all vertices $u$ of this replacement path we have $|u - v_i| \leq cD(A)$ and therefore $\min_{p \in A} |u - p| \leq |u - v_i| \leq cD(A)$. From the triangle inequality it follows

$$\max_{p \in A} |u - p| \leq D(A) + \min_{p \in A} |u - p| \leq D(A) + cD(A) = (c + 1)D(A) \ .$$

The second property follows from the fact that all new edges inserted in $P'$ lie inside a disk or sphere of radius $cD(A)$. ■

**Procedure contract** $(P = (v_1, \ldots, v_m)$ : path, $A$ : area)
  **begin**
    Let $v_i$ be the first vertex of $P$ in $A$
    Let $v_j$ be the last vertex of $P$ in $A$
    Let $P' = (w_1, \ldots, w_k)$ be a path between $v_i = w_1$
      and $v_j = w_k$ satisfying the weak spanner property
    **return** $(v_1, \ldots, v_{i-1}, w_1, \ldots, w_k, v_{j+1}, \ldots, v_m)$
  **end**

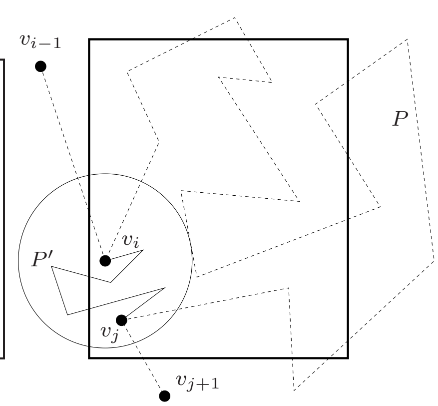



Figure 5: The **contract** procedure and an example for $\mathcal{D} = 2$

**Lemma 5.8** *For $D \geq 3(2c\sqrt{\mathcal{D}+3}+2)d$ the procedure $P' =$***clean-up**$(P, L, d, D)$ *satisfies the four properties power efficiency, locality, empty space, and continuity of long edges.*

1. **Locality** *For all vertices $u \in V(P')$ there exists $v \in V(P)$ such that*
$$|u - v| \;\leq\; (\sqrt{\mathcal{D}} + \sqrt{\mathcal{D}+2}) \cdot c \cdot d\;.$$

2. **Continuity of long edges** *For all edges $e \in E(P')$ with $|e| > 2c\sqrt{\mathcal{D}+3}d$ it holds $e \in E(P)$.*

3. **Power efficiency** *For all $k > 2c\sqrt{\mathcal{D}+3}$:*
$$\sum_{e \in E(P'): 2c\sqrt{\mathcal{D}+3}d < |e| \leq kd} |e|^{\mathcal{D}} \;\leq\; k^{\mathcal{D}}\, d^{\mathcal{D}}\, \#F(P, G_d)\;,$$
*where $\#F(P, G_d)$ denotes the number of grid cells of $G_d$ where at least one vertex of $P$ lies which is the end point of an edge of length at least $2c\sqrt{\mathcal{D}+3}d$.*

4. **Empty space** *For all grid cells $C$ of $G_D$ we have at least one $G_d$-sub-cell within $C$ without a vertex of $P'$.*

**Proof:** All cells of $G_d$ are called sub-cells in this proof for distinguishing them from the cells of $G_D$

Observe that the **clean-up** procedure uses only contract-operations to change the path. As parameters for this procedure we use either a grid sub-cell $C$ of edge length $d$ and diameter $D(C) = \sqrt{\mathcal{D}}d$ or two horizontally neighboring grid sub-cells $Z$ and $Z'$ with edge lengths $d$ with diameter $D(Z \cup Z') = \sqrt{\mathcal{D}+3}d$.

Further note that in the first loop each sub-cell $C$ of the grid $G_d$ will be treated by the contract-procedure once. The reason is that the contract procedures produce edges with lengths of at most $2c\sqrt{\mathcal{D}}d$, while each sub-cell will lose all but two edges of $P$ with minimum length greater than $2c\sqrt{\mathcal{D}}d$. This also proves that the first loop always halts.

Now consider the second while-loop and concentrate on the part inside the loop before the contract-operation takes place. Since in every sub-cell of $C$ we have a vertex of $P$ we can compute the ordering $Z_i$ and $Z'_i$ as described by the algorithm. The main observation is that until the first two neighboring sub-cells $Z$ and $Z'$ from these sets are found, no two sub-cells $Z$ and $Z'$ from $Z \in \{Z_j\}_{j \leq i}$ and $Z' \in \{Z'_j\}_{j \leq i}$ are horizontally neighboring. Considering only points of $P$ lying in these sub-cells implies that there is at least one empty sub-cell in $C$.

The situation changes slightly if we apply the contract-operation. Then an intermediate path will be added and possibly some of the empty sub-cells will start to contain vertices of the path. However, only sub-cells in a Euclidean distance of $c\sqrt{\mathcal{D}+3}d$ from sub-cells $Z$ and $Z'$ are affected by this operation. Now consider a square or cube $Q$ (see Figure 4) of $(2c\sqrt{\mathcal{D}+3}+2)d \times (2c\sqrt{\mathcal{D}+3}+2)d$ sub-cells in the middle of $C$. Then at least two horizontally neighboring sub-cells will not be influenced by this contract-operation and thus remain empty.

One cannot completely neglect the influence of this operation to a neighboring grid cell of $C$. However, since $D \geq 3(2c\sqrt{\mathcal{D}+3}+2)d$ the inner square or cube $Q$ is not affected by contract-operation in neighboring grid cells of $C$ because of the locality of the contract-operation.

This means if a cell $C$ was object to the second while-loop, then an empty sub-cell will be produced which remains empty for the rest of the procedure. Hence, the second loop also terminates. We now check the four required properties.

**Locality.** After the first loop the locality is satisfied even within a distance of $c\sqrt{\mathcal{D}}d$. For this, observe that all treated cells contain end points of edges longer than $2c\sqrt{\mathcal{D}}d$ which cannot be produced by contract-operations in this loop. Hence, if a cell is object to the contract-operation it was occupied by a vertex of $P$ from the beginning. Then from Lemma 5.7 it follows that for all new vertices on the path $P$ there exists at least one old vertex in distance $c\sqrt{\mathcal{D}}d$ after the first loop.

For the second loop we need to distinguish two cases. First, consider a cell $C$ where in the inner square or cube an empty sub-cell exists. In this case this cell will never be treated by this second loop. If new vertices are added to the path within this cell, then this will be caused by a contract-operation in a neighboring cell and will be considered in the second case.

Now consider all cells with preoccupied inner squares or cubes (preoccupation refers to the outcome of the first loop). These cells can be object to contract-operations of the second loop. However, they will add only vertices to their own sub-cells or to the outer sub-cells of neighboring cells. So, new vertices are added within a distance of $c\sqrt{\mathcal{D}+3}d$ of vertices in the path at the beginning of the second loop. As we have seen above every such vertex is only $c\sqrt{\mathcal{D}}d$ away from an original vertex of a path. This gives a locality of distance $(\sqrt{\mathcal{D}}+\sqrt{\mathcal{D}+3})cd$.

```
Procedure clean-up (P, L, d, D)
  begin
    while three edges exist in P longer than 2c√𝒟d starting or ending in the same cell of G_d
    do
      Let C be such a cell in G_d
      P ← contract(P, C)
    od
    while there exists a cell in G_D where at least one vertex of P is in each of its G_d-sub-cells
    do
      Let C be such a cell of G_D
      Let rank_P(u) be the position of a vertex u in P
      Sort all cells Z_1, ..., Z_{(D/d)^2} of G_d in C according to min_{u∈Z_i∩V(P)}{rank_P(u)}
      Sort all cells Z'_1, ..., Z'_{(D/d)^2} of G_d in C according to max_{u∈Z'_i∩V(P)}{rank_P(u)}
      i ← 1
      while cell Z_i is neither horizontally neighboring to one of the cells {Z'_1, ..., Z'_i}
        nor cell Z'_i is horizontally neighboring to one of the cells {Z_1, ..., Z_i}
        do
          i ← i + 1
        od
      Let z and z' be the two neighboring cells from {Z_1, ..., Z_i} and {Z'_1, ..., Z'_i}
      P ← contract(P, z ∪ z')
    od
    return P
  end
```

Figure 6: The **clean-up** procedure

**Continuity of long edges.** Since the parameterized areas or volumes for the contract operation have a maximum diameter of $\sqrt{\mathcal{D}+3}d$ this property follows directly from Lemma 5.7.

**Power efficiency.** After the first loop the number of edges longer than $2c\sqrt{\mathcal{D}}d$ is bounded by $\#F(P, G_d)$, because in every occupied sub-cell at most two edges start or end and each edge has two end points. Clearly, this number is an upper bound for edges longer than $2c\sqrt{\mathcal{D}+3}d$. In the second loop no edges longer than $2c\sqrt{\mathcal{D}+3}d$ will be added. This directly implies the wanted bound.

**Empty space.** As we have already pointed out the second loop always halts. Therefore the empty space property holds. ■

**Lemma 5.9** *Given a path $P_1$ with source $s$ and target $t$ such that $\forall u \in V(P_1) : |u-s| \le L$, where $L = c \cdot |s-t|$. Iteratively apply $P_{i+1} = \text{clean-up}(P_i, L_i, d_i, D_i)$ for $i = 1, 2, \ldots$ where $L_i = \sum_{j=1}^{i} D_j$, $D_i = L\beta^{1-i}$, $d_i = L\beta^{-i}$ for $\beta = 3(2c\sqrt{\mathcal{D}+3}+2)$. Then $P_m$ for $m = \max\{1, \lceil \log_{\beta^{-1}}(\min_{u,v\in V}|u-v|)/L\rceil\}$ is a path connecting $s$ to $t$ obeying the $\left(\mathcal{O}(c^{4\mathcal{D}}), \mathcal{D}\right)$-power spanner property.*

**Proof:** For this proof we make use of the four properties of the clean-up procedure. First note that the square or cube of edge length $L_i$ containing all vertices of path $P_i$ can increase (see Figure 7). However we can bound this effect by the locality property, giving $L_{i+1} \le L_i + 2(\sqrt{\mathcal{D}}+\sqrt{\mathcal{D}+3})cd_i$, where $d_i = L\cdot\beta^{-i}$. Since, $2(\sqrt{\mathcal{D}}+\sqrt{\mathcal{D}+3})c \le 6c\sqrt{\mathcal{D}+3}+6 = \beta$ for $c \ge 1$, we get $L_{i+1} \le L_i + D_i$. Hence, our choice of

$L_i$ fulfills the requirements and we get an upper bound of $L_i = L_{i-1} + D_{i-1} \leq L + \sum_{j=2}^{\infty} D_j \leq 2L$ for all $i > 1$.

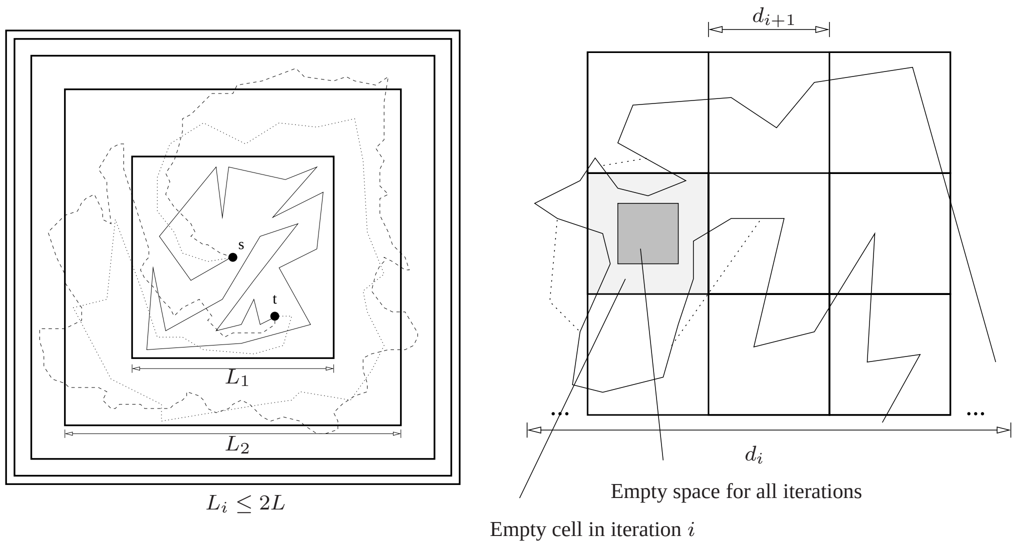



Figure 7: Increase in path and lower bound for empty space for $\mathcal{D} = 2$

Let $F_i = \#F(P_i, G_{d_i})$. Then $A_i = (d_i)^{\mathcal{D}} F_i$ denotes the area or volume of all grid cells in $G_{d_i}$ with a vertex of the path $P_i$ which is the end point of an edge with length of at least $2c\sqrt{\mathcal{D}+3}d_i$. In the next iteration near the middle in each of these cells an empty space will be generated with an area or volume of $(d_{i+1})^{\mathcal{D}}$ (see Figure 7). Because of the locality property at most the following term of the side length of this area or volume is subtracted

$$\sum_{j=i+2}^{\infty} (\sqrt{\mathcal{D}} + \sqrt{\mathcal{D}+3})cd_j \leq \sum_{j=i+2}^{\infty} \frac{\beta(\sqrt{\mathcal{D}} + \sqrt{\mathcal{D}+3})cd_j}{\beta} < \frac{1}{3} \sum_{j=i+2}^{\infty} d_{j-1} \leq \frac{1}{3} \sum_{j=i+1}^{\infty} d_j \leq \frac{1}{3} d_{i+1} \, .$$

Hence, an empty area or volume of at least $(\frac{d_{i+1}}{3})^{\mathcal{D}}$ remains after applying all clean-up procedures. Let $E_i$ be the sum of all these areas or volumes in this iteration, then $E_i \geq F_i \cdot (\frac{d_{i+1}}{3})^{\mathcal{D}} = F_i \cdot (\frac{d_i}{3})^{\mathcal{D}} \cdot \frac{1}{\beta^{\mathcal{D}}}$. Therefore we have $A_i \leq 3^{\mathcal{D}} \beta^{\mathcal{D}} E_i$. Clearly, these empty areas or volumes in this iteration do not intersect with empty areas or volumes in other iterations (since they arise in areas or volumes which were not emptied before). Therefore all these spaces are inside the all-covering square or cube of side length $2L$ yielding $\sum_{i=1}^{\infty} E_i \leq 2^{\mathcal{D}} L^{\mathcal{D}}$.

Because of the long edge continuity property, edges of minimum length $2c\sqrt{\mathcal{D}+3}d_i$ do not appear in rounds later than $i$. Therefore, the following sum $S$ gives an upper bound on the power of the constructed path.

$$S = \sum_{i=1}^{\infty} \sum_{\substack{e \in E(P_i): \\ 2c\sqrt{\mathcal{D}+3}d_i \leq |e| < 2c\sqrt{\mathcal{D}+3}\beta d_i}} |e|^{\mathcal{D}} \, .$$

Now from the power efficiency property it follows

$$\begin{aligned}
S &\leq \sum_{i=1}^{\infty} 2^{\mathcal{D}} c^{\mathcal{D}} (\mathcal{D}+3)^{\mathcal{D}/2} \beta^{\mathcal{D}} (d_i)^{\mathcal{D}} \# F(P_i, G_{d_i}) \\
&= 2^{\mathcal{D}} c^{\mathcal{D}} (\mathcal{D}+3)^{\mathcal{D}/2} \beta^{\mathcal{D}} \sum_{i=1}^{\infty} (d_i)^{\mathcal{D}} F_i \\
&= 2^{\mathcal{D}} c^{\mathcal{D}} (\mathcal{D}+3)^{\mathcal{D}/2} \beta^{\mathcal{D}} \sum_{i=1}^{\infty} A_i \\
&\leq 2^{\mathcal{D}} 3^{\mathcal{D}} c^2 (\mathcal{D}+3)^{\mathcal{D}/2} \beta^{2\mathcal{D}} \sum_{i=1}^{\infty} E_i \\
&\leq 2^{\mathcal{D}} 3^{\mathcal{D}} c^{\mathcal{D}} (\mathcal{D}+3)^{\mathcal{D}/2} \beta^{2\mathcal{D}} L^{\mathcal{D}} \\
&\leq 2^{\mathcal{D}} 3^{\mathcal{D}} c^{\mathcal{D}} (\mathcal{D}+3)^{\mathcal{D}/2} \beta^{2\mathcal{D}} (|s-t|)^{\mathcal{D}} = \mathcal{O}(c^{4\mathcal{D}} (|s-t|)^{\mathcal{D}}) \, .
\end{aligned}$$

■

This lemma completes the proof of the theorem. ■

## 5.3 Weak Spanners are not always Power Spanners for $\delta < \mathcal{D}$

In this subsection we show that there exist weak $c$-spanners that are not $(C, \delta)$-power spanners for any constant $C$ and $\delta < \mathcal{D}$. We introduce a new fractal curve that is similar to the HILBERT Curve (see Figure 10) to prove this claim.

**Theorem 5.10** *To any $\delta < \mathcal{D}$, there exists a family of geometric graphs $\mathcal{G} = (V, E)$ with $V \subset \mathbb{R}^{\mathcal{D}}$ which are weak $c$-spanners for a constant $c$ but not $(C, \delta)$-power spanners for any fixed $C$.*

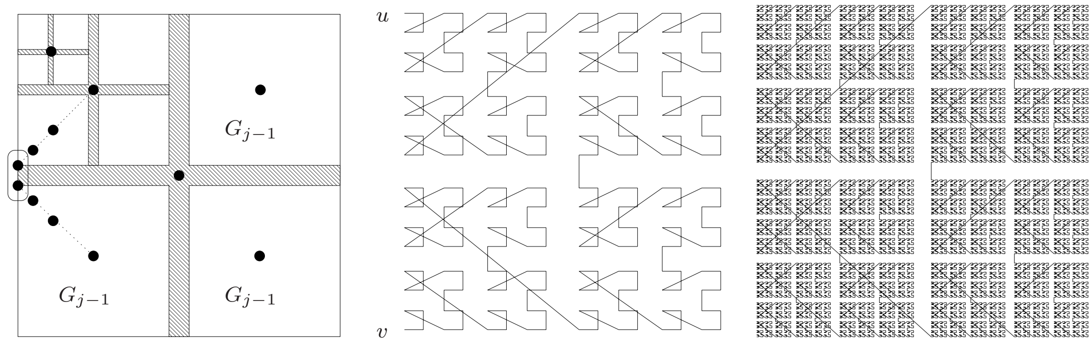
**Proof:** As $\delta < \mathcal{D}$, there is a $k \in \mathbb{R}$ such that $2 < k < (2^{\mathcal{D}})^{1/\delta}$. We present a recursive construction (see Figure 8 and Figure 9). Fix $u^1 = (1/2, \ldots, 1/2) \in \mathbb{R}^{\mathcal{D}}$. In each following recursion step $j$, we replace every existing vertex $u^i = (u^i_1, \ldots, u^i_{\mathcal{D}})$ by $2^{\mathcal{D}}$ new vertices $u^{2^{\mathcal{D}}(i-1)+1}, \ldots, u^{2^{\mathcal{D}}(i-1)+2^{\mathcal{D}}}$ with $u^{2^{\mathcal{D}}(i-1)+r}(s) = u^i_s \otimes_{\mathcal{D},r,s} d$ where $\otimes_{\mathcal{D},r,s}$ defines the operator $+$ or $-$ with $\otimes_{2,1,1} = -$, $\otimes_{2,1,2} = +$, $\otimes_{2,2,1} = +$, $\otimes_{2,2,2} = +$, $\otimes_{2,3,1} = +$, $\otimes_{2,3,2} = -$, $\otimes_{2,4,1} = -$, $\otimes_{2,4,2} = -$,

$$\otimes_{\mathcal{D},r,s} = \begin{cases} \otimes_{\mathcal{D}-1,r,s} & : \; r = \{1, \ldots, 2^{\mathcal{D}-1}\}, s < \mathcal{D} \\ \otimes_{\mathcal{D}-1,2^{\mathcal{D}}+1-r,s} & : \; r = \{2^{\mathcal{D}-1}+1, \ldots, 2^{\mathcal{D}}\}, s < \mathcal{D} \\ + & : \; r = \{1, \ldots, 2^{\mathcal{D}-1}\}, s = \mathcal{D} \\ - & : \; r = \{2^{\mathcal{D}-1}+1, \ldots, 2^{\mathcal{D}}\}, s = \mathcal{D} \end{cases} ,$$

and $d := 1/(2k^j)$. Finally, we consider the graph $G_j := (V_j, E_j)$ with $V_j := \{u^i \mid i \in \{1, \ldots, (2^{\mathcal{D}})^j\}\}$ and $E_j := \{(u^i, u^{i+1}) \mid i \in \{1, \ldots, (2^{\mathcal{D}})^j - 1\}\}$. The resulting graph for $\mathcal{D} = 2$ after $4$ recursion steps with $k = 2.1$ is given in Figure 8(b) (see also Figure 9 for $\mathcal{D} = 3$). Let $u = u^1$ and $v = u^{4^j}$.

**Lemma 5.11** *The graph $G_j$ is a weak $c$-spanner for $c := \frac{\sqrt{\mathcal{D}}k(k-1)}{k-2}$ independent of $j$.*

**Proof:** We prove the claim by induction over $j$. For $j = 1$ the weak stretch factor is dominated by the path between $u$ and $v$. The distance between $u$ and $v$ is $1/k$. The farthest vertex on the path from $u$ to $v$ is $u^3$. It holds that $|u - u^3| \leq \sqrt{\mathcal{D}}/k$. Hence, we get the weak stretch factor $\sqrt{\mathcal{D}} \leq \frac{\sqrt{\mathcal{D}}k(k-1)}{k-2} = c$. Now we consider $G_j$ for any $j$. We can divide the graph $G_j$ into four parts $G^1_j, \ldots, G^4_j$. By the definition of our recursive construction each part equals the graph $G_{j-1}$. For two vertices in one part the required weak $c$-spanner property holds by induction. We have to concentrate on two vertices which are chosen from two different parts. Since $G^i_j$ is connected to $G^{i+1}_j$ it is sufficient to consider a vertex from $G^1_j$ and a vertex




(a) Idea (b) After 4 steps (c) After 7 steps

Figure 8: Recursive construction for $\mathcal{D} = 2$: the underlying idea and two examples for $k = 2.1$

from $G_j^4$. On the one hand, the weak stretch factor is affected by the shortest distance between such chosen vertices. On the other hand, this distance is given by (see also Figure 8(a))

$$\left(\frac{1}{2} \cdot (1 + \frac{1}{k} - \sum_{i=2}^{j} (\tfrac{1}{k})^i) - \frac{1}{2}\right) \cdot 2 = \frac{1}{k} - \sum_{i=2}^{j} (\tfrac{1}{k})^i \geq \frac{k-2}{k(k-1)}$$

The entire construction lies in a bounded square or cube of side length $1$, and hence we get a weak stretch factor of at most $\frac{\sqrt{\mathcal{D}}k(k-1)}{k-2} = c$. ■

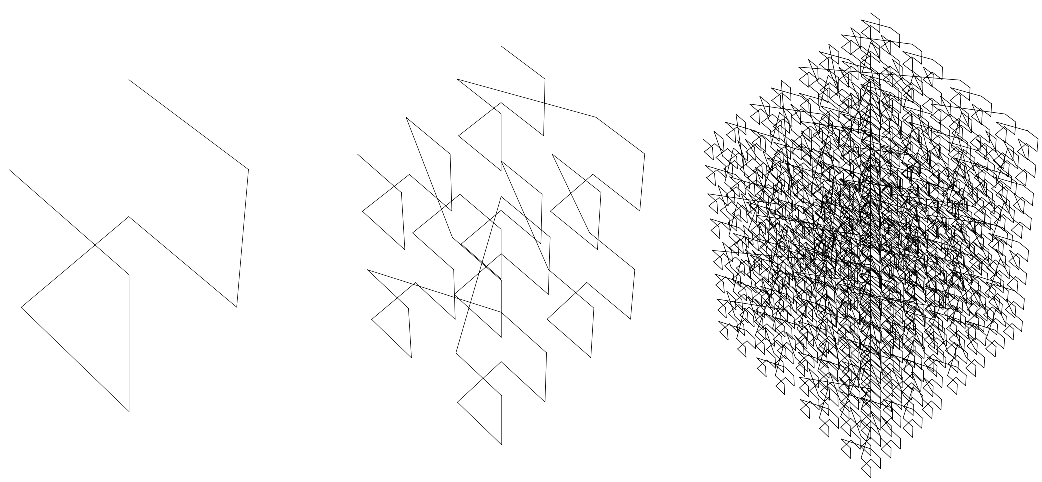


Figure 9: Our recursive construction for $\mathcal{D} = 3$ and $k = 2.1$

**Lemma 5.12** *The graphs $G_j$ are not $(C, \delta)$-power spanners for any fixed $C$.*

**Proof:** It suffices to consider the $\delta$-cost of the path from $u$ to $v$. The direct link from $u$ to $v$ has $\delta$-cost at most $1$. For any path $P$ from $u$ to $v$ in $G$, it holds that

$$\|P\|^\delta \quad \geq \quad (2^{\mathcal{D}} - 1) \cdot (2^{\mathcal{D}})^j \cdot \left((\tfrac{1}{k})^j\right)^\delta \quad = \quad (2^{\mathcal{D}} - 1) \cdot \left(\frac{\mathcal{D}}{k^\delta}\right)^j$$

which goes to infinity if $j \to \infty$ for $k < 2^{\mathcal{D}/\delta}$. ■

Combining Lemma 5.11 and Lemma 5.12 proves Theorem 5.10. ■

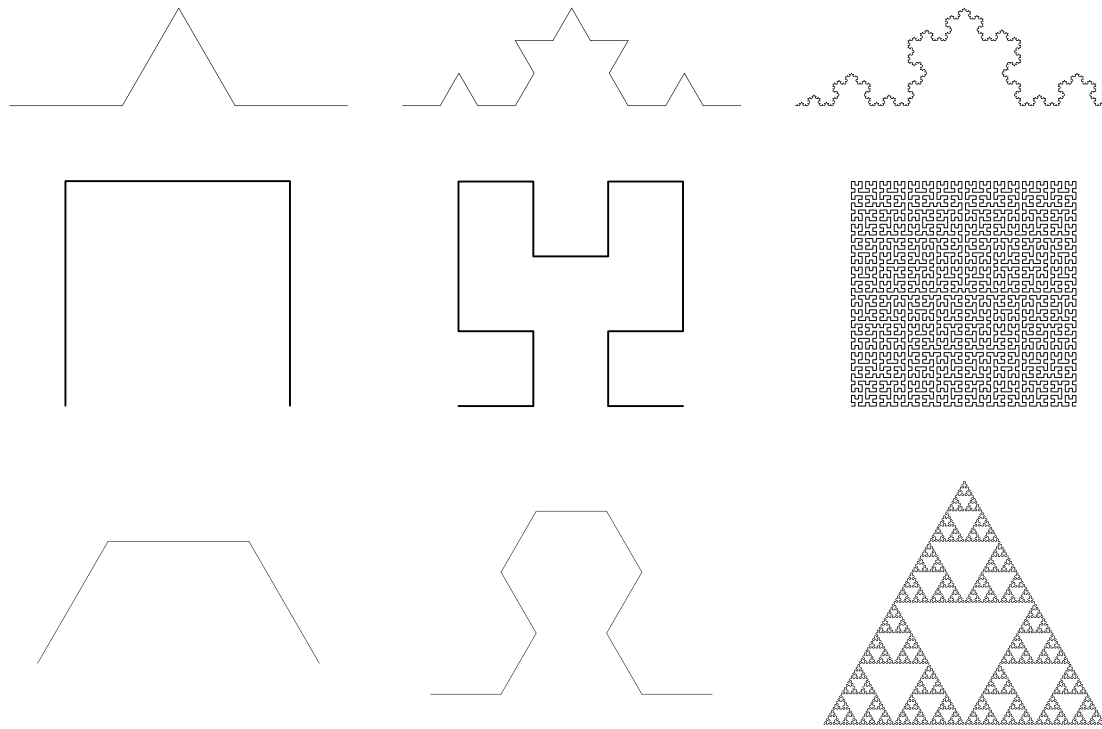


Figure 10: Three generators and the fractal curves they induce due to KOCH, HILBERT and SIERPINSKI

## 5.4 Fractal Dimension

In this subsection we generalize the analysis used in Lemma 5.12. For this purpose we consider a self-similar polygonal fractal curve $\Gamma$ as the result of repeated application of some generator $K$ being a polygonal chain with starting point $u$ and end point $v$. This is illustrated in Figure 2 showing a generator (left) and the resulting fractal curve (right part); see also [5, 22]. But there are plenty of other examples: the KOCH Snowflake, SIERPINSKI's Arrowhead Curve or the space filling HILBERT Curve (see Figure 10). Recall that the fractal dimension of $\Gamma$ is defined as

$$\frac{\log(\text{number of self-similar pieces})}{\log(\text{magnification factor})}$$

**Theorem 5.13** *Let $K$ be a polygonal chain, $\Gamma_n$ the result of $n$-fold application of $K$, and $\Gamma$ the final self-similar polygonal fractal curve with dimension $\mathcal{D}_f$. Then, for all $\delta < \mathcal{D}_f$, there is no fixed $C$ such that $\Gamma_n$ is a $(C, \delta)$-power spanner for all $n$.*

**Proof:** Let $p$ denote the number of self-similar pieces in $\Gamma_n$ and $m$ the magnification factor. Then by definition, we have $\mathcal{D}_f = \log(p)/\log(m)$. Now consider the $\delta$-cost of the (unique) path $P$ in $\Gamma_n$ from $u$ to $v$. Since $\Gamma_n$ is constructed recursively we get in the $n$-th step:

$$\|P\|^\delta \quad = \quad p^n \cdot \left(\left(\tfrac{1}{m}\right)^n\right)^\delta \quad = \quad \left(\frac{p}{m^\delta}\right)^n$$

Note that $\|P\|^\delta$ is unbounded iff $p/m^\delta > 1$, that is, iff $\delta < \log(p)/\log(m) = \mathcal{D}_f$. ■

The fractal dimensions of KOCH's, SIERPINSKI's and HILBERT's Curves are well-known. Therefore by virtue of Theorem 5.13, the KOCH Curve is not a $(c, \delta)$-power spanner for any $c$ and $\delta < \log(4)/\log(3) \approx 1.26$; similarly, SIERPINSKI's Arrowhead Curve is not a $(c, \delta)$-power spanner for any $c$ and $\delta < \log(3)/\log(2) \approx 1.58$; and HILBERT's Curve is not a $(c, \delta)$-power spanner for any $c$ and $\delta < 2$. One can show that KOCH's Curve is a weak spanner (the proof is analogously to Theorem 3.1). However, both SIERPINSKI's and HILBERT's Curves are not weak spanners as their inner vertices come arbitrarily close to each other. Further examples for self-similar polygonal curves can be found in [5, 22].

## 6 Power Spanner Hierarchy

In the following we show that for $\Delta > \delta > 0$, a $(c, \delta)$-power spanner is also a $(C, \Delta)$-power spanner with $C$ depending only on $c$ and $\Delta/\delta$. Then we show that the converse is not true, in general, by presenting to each $\Delta > \delta > 0$ a family of graphs which are $(c, \Delta)$-power spanners for some constant $c$ but not $(C, \delta)$-power spanners for any fixed $C$.

**Theorem 6.1** *Let $G = (V, E)$ be a $(c, \delta)$-power spanner with $V \subset \mathbb{R}^{\mathcal{D}}$, $0 < \delta < \Delta$. Then $G$ is also a $(C, \Delta)$-power spanner for $C := c^{\Delta/\delta}$.*

**Proof:** Let $u, v \in V$ be two arbitrary vertices. Since $G$ is a $(c, \delta)$-power spanner there exists a path $P = (u = u_1, \ldots, u_\ell = v)$ with $\|P\|^\delta = \sum_{i=1}^{\ell-1} |u_i - u_{i+1}|^\delta \leq c \cdot |u - v|^\delta$. The function $f(x) = x^{\Delta/\delta}$ is convex on $[0, \infty[$, hence we can apply JENSEN's inequality and get

$$\|P\|^\Delta = \sum_{i=1}^{\ell-1} |u_i - u_{i+1}|^\Delta = \sum_{i=1}^{\ell-1} \left(|u_i - u_{i+1}|^\delta\right)^{\Delta/\delta} \leq \left(\sum_{i=1}^{\ell-1} |u_i - u_{i+1}|^\delta\right)^{\Delta/\delta} \leq c^{\Delta/\delta} \cdot |u - v|^\Delta .$$

■

**Theorem 6.2** *Let $0 < \delta < \Delta$. There is a family of geometric graphs which are $(c, \Delta)$-power spanners but not $(C, \delta)$-power spanners for any fixed $C$.*

**Proof:** We slightly modify the construction from the proof of Theorem 5.1 by placing $n$ vertices $u = u_1, \ldots, u_n = v$ on an appropriately scaled circle such that the Euclidean distance between $u$ and $v$ is 1 and $|v_i - v_{i+1}| = (1/i)^{1/\delta}$ for all $i = 1, \ldots, n-1$. Now in the graph $G = (V, E)$ with edges $(v_i, v_{i+1})$, the unique path $P$ from $u$ to $v$ has $\Delta$-cost

$$\|P\|^\Delta \quad = \quad \sum_{i=1}^{n-1} (1/i)^{\Delta/\delta} \quad \leq \quad \sum_{i=1}^{\infty} (1/i)^{\Delta/\delta} \quad =: \quad c$$

a convergent series since $\Delta/\delta > 1$. This has to be compared to the $\Delta$-cost and/or to the $\delta$-cost of the direct link from $u$ to $v$ which amount both to $1$. On the other hand, the $\delta$-cost of $P$ is given by the harmonic series $\sum_{i=1}^{n-1} (1/i)^{\delta/\delta} = \Theta(\log n)$ and thus cannot be bounded by any constant $C$. ■

## 7 Applications in Wireless Networks

In this section we apply our results to topology control in wireless networks. Especially, we consider the well-known sparsified Yao-graph (SparsY-graph or YY) introduced in parallel in [10, 15, 26] and show that this graph is a power spanner. In the following we improve results presented in [11]. First, we need a formal definition.

**Definition 7.1** *Let $V \subset \mathbb{R}^2$, $k \in \mathbb{N}$ and $G = (V, E)$ be a geometric graph. The area around a node $u \in V$ is divided into $k$ non-overlapping sectors or cones of angle $\theta = 2\pi/k$. We denote the sector of $u$ in which a node $v \in V$ lies by $\sphericalangle(u, v)$.*

- *$G$ is the* Yao-graph *of $V$, if $E := \{(u, v) \mid \forall w \neq u : \sphericalangle(u, v) = \sphericalangle(u, w) \Rightarrow |u - v| < |u - w|\}$.*
- *$G$ is the* SparsY-graph *of $V$, if $E := \{(u, v) \in E(\overline{G}) \mid \forall w \neq u : ((w, v) \in E(\overline{G})$ and $\sphericalangle(v, w) = \sphericalangle(v, u)) \Rightarrow |u - v| < |w - v|\}$ where $\overline{G}$ denotes the Yao-graph of $V$.*
- *$G$ is the* SymmY-graph *of $V$, if $E := \{(u, v) \in E(\overline{G}) \mid (v, u) \in E(\overline{G})\}$ where $\overline{G}$ denotes the SparsY-graph of $V$.*

A lot is known about the Yao-graph and its variants, an overview is given in [24]. In the SparsY-graph every node tries to build up an edge (or a connection) to the nearest neighbor in each sector, but a node accepts only the shortest ingoing edge (or connection) in each sector. It is an open problem wether all SparsY-graphs are $c$-spanners for a constant $c$. In [11] it was shown that all SparsY-graphs are $c$-power spanners for $k \geq 120$. We show that all SparsY-graphs are $c$-power spanners for $k > 6$.

**Lemma 7.2** *Let $V \subset \mathbb{R}^2$ and $k > 6$. Consider two vertices $u, v$, then for all $w \in V$ with $|u - w| \leq |u - v|$ and $\sphericalangle(u, w) = \sphericalangle(u, v)$ it holds that $|u - w| + |w - v| < |u - v| + 2\sin(\pi/k)|u - v|$.*

**Proof:** Let $w'$ be the point on the line segment $uv$ with $|u - w'| = |u - w|$. Then $|w' - v| = |u - v| - |u - w'| = |u - v| - |u - w|$ and therefore $|u - w| = |u - v| - |w' - v|$. Furthermore, we have $|w - w'| < 2\sin(\pi/k)|u - v|$ and $|w - v| \leq |w - w'| + |w' - v|$. Combining the facts yields the claim. ∎

First, we prove that all SparsY-graphs are weak $c$-spanners when more than 6 sectors per node are used.

**Theorem 7.3** *Let $V \subset \mathbb{R}^2$ and $k > 6$. Then all SparsY-graphs are weak $c$-spanners for $c = \frac{1}{1 - 2\sin(\pi/k)}$.*

**Proof:** Let $G = (V, E)$ be the SparsY-graph and $G_Y = (V, E_Y)$ be the underlying Yao-graph. For any two vertices $u, v \in V$ we will show how to find a directed path from $u$ to $v$ in the SparsY-graph that is inside a disk with center $u$ and radius $|u - v|/(1 - 2\sin(\pi/k))$.

For a sector $i$, define the Yao-neighbor $w$ of the vertex $u$ as the (unique) vertex $w$ with $(u, w) \in E_Y$. Now if $u$ has no directed edge in a sector $i$ in $G$, then either the sector is empty, i.e. there is no edge in the Yao-graph, or there is a Yao-neighbor $w$ in sector $i$, i.e. $(u, w) \in E_Y$. In the second case there must be another vertex $w'$ in another sector of $u$, but in the same sector of $w$ as $u$, with $(w', w) \in E$ and $|w' - w| < |u - w|$ and $|u - w'| < |u - w|$ since $k > 6$, see Figure 11 (left). Note that $u$ has at least one nearest neighbor $w''$ in a sector, i.e. $\exists w'' \in V \;:\; (u, w'') \in E$. This is the vertex $w'' \in V$ with the shortest Euclidean distance to $u$.

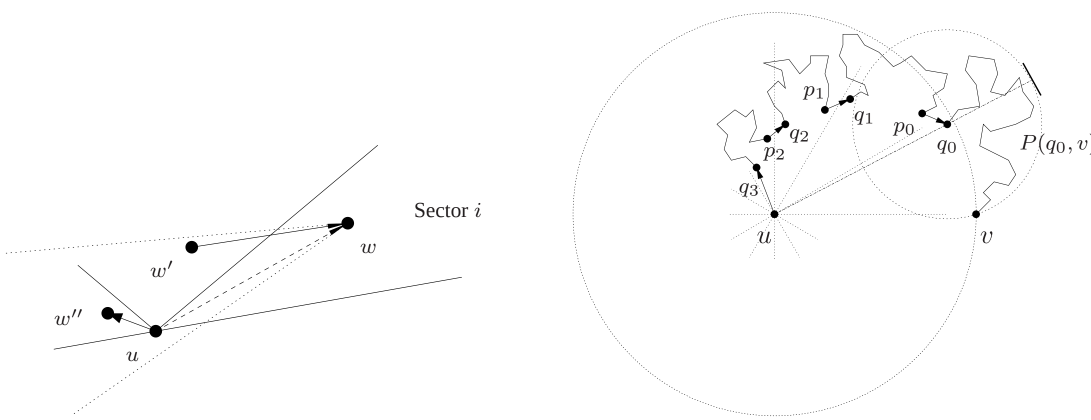



Figure 11: Weak spanner property of the SparsY-graph

Now we recursively construct the path $P(u, v)$ from $u$ to $v$ using some of the Yao-neighbors of $u$. An example of such a path is given in Figure 11 (right). If $(u, v) \in E$ then $P(u, v) = (u, v)$, if $u = v$ then $P(u, v) = (u = v)$. If in sector $i = \sphericalangle(u, v)$ the Yao-neighbor, called $q_0$, is not directly connected to $u$. Then we know that there exists an edge $(p_0, q_0) \in E$, where $p_0$ is in a sector $i_1 \neq i_0$ of $u$ and $|u - p_0| < |u - q_0|$ since $k > 6$. Furthermore, we have that $|u - q_0| \leq |u - v|$. Then we repeat this consideration for the sector $i_1$ and replace $p_0$ by $v$. This iteration ends when a Yao-neighbor $q_m$ or $p_m$ is directly connected to $u$, i.e. $(u, q_m) \in E$ or $(u, p_m) \in E$. Because every node has at least one neighbor in $E$ this process terminates.

Now we recursively define the path $P(u, v)$ from $u$ to $v$ that terminates at node $q_m$ by

$$P(u, v) = (u, q_m) \circ P(q_m, p_{m-1}) \circ (p_{m-1}, q_{m-1}) \circ \ldots \circ P(q_1, p_0) \circ (p_0, q_0) \circ P(q_0, v).$$

For $p_m$ the path can be defined analogously: we replace $(u, q_m)$ by $(u, p_m) \circ (p_m, q_m)$. Note that all nodes $p_i, q_i$ are inside the disk with center $u$ and radius $|u - v|$, see Figure 11 (right). Furthermore, we have $|q_i - p_{i-1}| < |u - v|$. In the next recursion step, i.e. if we assume that the same construction works for all $P(q_i, p_{i-1})$, vertices of the path may lie outside of this disk. However Lemma 7.2 implies that the maximum disk amplification of this recursion step can be bounded by $2\sin(\frac{\pi}{k})\ |u - v|$. Now let $r$ be the depth of the recursion, then by

$$\sum_{i=0}^{r} \left(2\sin\left(\frac{\pi}{k}\right)\right)^i\ |u - v| \leq \frac{1}{(1 - 2\sin(\frac{\pi}{k}))} \cdot |u - v|$$

it follows, that the path $P(u, v)$ from $u$ to $v$ lies completely inside the disk with center $u$ and radius $|u - v|/(1 - 2\sin(\frac{\pi}{k}))$. ■

Now, we can apply our results on the relation between weak spanners and power spanners and get the following corollary.

**Corollary 7.4** *Let $V \subset \mathbb{R}^2$ and $k > 6$. Then all SparsY-graphs are $c$-power spanners for a constant $c$.*

**Proof:** We combine Theorem 7.3 and Theorem 5.6 and the claim follows directly. ■

# 8 Conclusions

In this work we investigated the relations between spanners, weak spanners, and power spanners for $V \subset \mathbb{R}^{\mathcal{D}}$ for any constant $\mathcal{D}$. The results are summarized in Table 1. An entry in the table should be read from left to right downwards, e.g., every $c$-spanner is a $(c^\delta, \delta)$-power spanner). For $\delta \geq \mathcal{D}$ it turns out that being a spanner is the strongest property, followed by being a weak spanner and finally being a $(c, \delta)$-power spanner. For $1 < \delta < \mathcal{D}$, spanner is still strongest whereas weak spanner and $(c, \delta)$-power spanner are not related to each other. For $0 < \delta < 1$ finally, $(c, \delta)$-power spanners are both spanners and weak spanners. All stretch factors in these relations are constant and are pairwise polynomially bounded.

| | | | |
|---|---|---|---|
| **$c$-spanner** | $c$ | $c$ | $(c^\delta, \delta)$ |
| **weak $c$-spanner** | (unbounded) | $c$ | $\left(\mathcal{O}\left(c^{2\mathcal{D}+\epsilon}/(1 - 2^{-\epsilon})\right), \mathcal{D} + \epsilon\right)$<br>$\left(\mathcal{O}(c^{4\mathcal{D}}), \mathcal{D}\right)$<br>$\left(\text{unbounded}, \mathcal{D} - \epsilon\right)$ |
| **$(c, \delta)$-power spanner** | (unbounded) | (unbounded) | for $\Delta > \delta$: $(c^{\Delta/\delta}, \Delta)$<br>for $\Delta < \delta$: (unbounded, $\Delta$) |
| *is a* | **-spanner** | **-weak spanner** | **-power spanner** |

Table 1: Our results on the relations between spanners, weak spanners and power spanners

Although our results are exhaustive with respect to the different kinds of geometric graphs and in terms of $\delta$, one might wonder about the optimality of the bounds obtained for $C$'s dependence on $c$; for instance: any $c$-spanner is a $(C, \delta)$-power spanner for $C = c^\delta$, $\delta > 1$ ; and this bound is optimal. But is there some $C = o(c^{2\mathcal{D}})$ such that any weak $c$-spanner is a $(C, \delta)$-power spanner as long as $\delta > \mathcal{D}$ ? Is there some $C = o(c^{4\mathcal{D}})$ such that any weak $c$-spanner is a $(C, \delta)$-power spanner ?

Simulations [15,23,26] indicate that the Yao-graph as well as its variants are spanners on random vertex sets in which the vertices are placed uniformly at random. We used our results on the relations between spanners, weak spanners, and power spanners to show that the SparsY-graph is already a power spanner when more than 6 sectors are available per node. In [16] we have shown that weak spanners are also suitable for congestion optimal wireless network topologies and hence we have another application.

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