Algorithm Theory 2008/09

Poposal for a solution of exercise sheet 3

Nov 12, 2008

TASK 1

1. Specify all primitive 8-th roots of unity.

   From the cancellation lemma follows that $\{\omega_8^0, \omega_8^2, \omega_8^4, \omega_8^6\}$ equals the 4-th roots of unity. Therefore they are not primitive 8-th roots of unity. The other roots $\{\omega_8^1, \omega_8^3, \omega_8^5, \omega_8^7\}$ are primitive as the following table shows.

x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
$\omega_8^1 \neq 1$	$\omega_8^2 \neq 1$	$\omega_8^3 \neq 1$	$\omega_8^4 \neq 1$	$\omega_8^5 \neq 1$	$\omega_8^6 \neq 1$	$\omega_8^7 \neq 1$	$\omega_8^8 = \omega_8^0 = 1$
ω_8^3	ω_8^6	ω_8^1	ω_8^4	ω_8^7	ω_8^2	ω_8^5	$\omega_8^8 = \omega_8^0 = 1$
ω_8^5	ω_8^2	ω_8^7	ω_8^4	ω_8^1	ω_8^6	ω_8^3	$\omega_8^8 = \omega_8^0 = 1$
ω_8^7	ω_8^6	ω_8^5	ω_8^4	ω_8^3	ω_8^2	ω_8^1	$\omega_8^8 = \omega_8^0 = 1$

2. Calculate the product of $p(x) = 3x - 1$ and $q(x) = 2x + 5$.

   (a) Compute the FFT of $p(x)$ and $q(x)$.

   $\boxed{\text{DFT}_4(p)}$

   $FFT([-1, 3], 2)$ for $n = 4$

   $d^{[0]} = FFT(1, 1) = -1$

   $d^{[1]} = FFT(3, 1) = 3$

   $\omega = 1, k = 0$

   $d_0 = d^{[0]} + \omega d^{[1]} = -1 + 3 = 2$

   $d_2 = d^{[0]} - \omega d^{[1]} = -1 - 3 = -4$

   $\omega = e^{2\pi i/4} \cdot 1 = i, k = 1$

   $d_1 = d^{[0]} + \omega d^{[1]} = -1 + 3i$

   $d_3 = d^{[0]} - \omega d^{[1]} = -1 - 3i$

$\boxed{\text{DFT}_4(q)}$

$FFT([2,5],2)$ for $n=4$
$d^{[0]} = FFT(5,1) = 5$
$d^{[1]} = FFT(2,1) = 2$
$\omega = 1, k = 0$
$d_0 = d^{[0]} + \omega d^{[1]} = 5 + 2 = 7$
$d_2 = d^{[0]} + \omega d^{[1]} = 5 - 2 = 3$
$\omega = e^{2\pi i/4} \cdot 1 = i, k = 1$
$d_1 = d^{[0]} + \omega d^{[1]} = 5 + 2i$
$d_3 = d^{[0]} + \omega d^{[1]} = 5 - 2i$

We calculate DFT_4, because pq is of degree 4.

(b) Give the point-value representation of pq at the k-th roots of unity for an appropriate choice of k.

		$\text{DFT}_4(p)$	$\text{DFT}_4(q)$	
i	ω_4^i	$p(\omega_4^i)$	$q(\omega_4^i)$	$pq(\omega_4^i)$
0	1	2	7	14
1	i	$-1+3i$	$5+2i$	$-11+13i$
2	-1	-4	3	-12
3	$-i$	$-1-3i$	$5-2i$	$-11-13i$

Point value representation:
$\{(1,14),(i,-11+13i),(-1,-12),(-i,-11-13i)\}$

(c) Compute the interpolation by using the FFT algorithm

$$a = \frac{1}{n}\left(r(\omega_4^4), r(\omega_4^3), r(\omega_4^2), r(\omega_4^1)\right)$$
$$= \frac{1}{n}\left(r(1), r(\omega_4^3), r(\omega_4^2), r(\omega_4^1)\right)$$
$$y = [pq(\omega_4^0), pq(\omega_4^3), pq(\omega_4^2), pq(\omega_4^1)] = [14, -11-13i, -12, -11+13i]$$

Note, that we have to re-order the input for the FFT.

$$
\begin{aligned}
d^{[0]} &= FFT([14,-12],2) \\
&= (\, FFT([14],1) + FFT([-12],1)\,,\ FFT([14],1) - FFT([-12],1)\,) \\
&= (2,26)
\end{aligned}
$$

$$
\begin{aligned}
d^{[1]} &= FFT([-11-13i,-11+13i],2) \\
&= (\, FFT([-11-13i],1) + FFT([-11+13i],1)\,,\ ...\,) \\
&= (-22,-26i)
\end{aligned}
$$

$\omega = 1$, $k = 0$
$d_0 = d_0^{[0]} + \omega d_0^{[1]} = 2 + 1 \cdot (-22) = -20$
$d_2 = d_0^{[0]} - \omega d_0^{[1]} = 2 - 1 \cdot (-22) = 24$
$\omega = e^{2\pi i/4} \cdot 1 = i$, $k = 1$
$d_1 = d_1^{[0]} + \omega d_1^{[1]} = 26 + i \cdot (-26i) = 52$
$d_3 = d_1^{[0]} - \omega d_1^{[1]} = 26 - i \cdot (-26i) = 0$

$DFT_4(r) = (-20,52,24,0)$
$a = \frac{1}{4} DFT_4(r) = (-5,13,6,0)$
$pq(x) = -5 + 13x + 6x^2 + 0x^3$

(d) Check the correctness of the result.
$pq(x) = (3x-1)(2x+5) = 6x^2 + 13x - 5$