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Algorithm Theory

4 Randomized Algorithms: Quicksort

Christian Schindelhauer

Albert-Ludwigs-Universität Freiburg
Institut für Informatik
Rechnernetze und Telematik
Wintersemester 2007/08

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Randomized algorithms

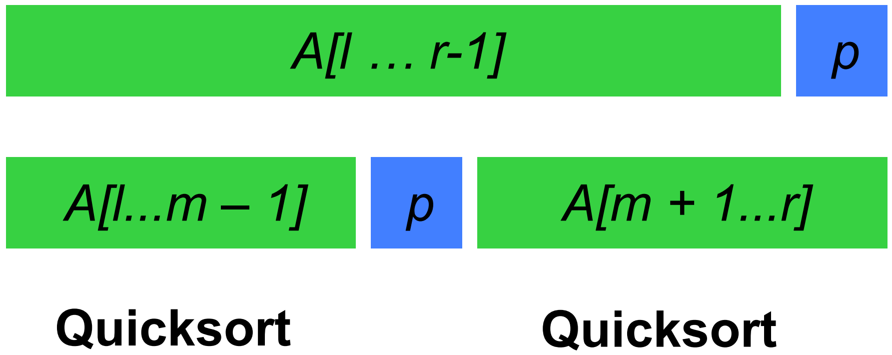
- Classes of randomized algorithms
- Randomized Quicksort
- Randomized algorithm for Closest Pair
- Randomized primality test
- Cryptography

Classes of randomized algorithms

- **Las Vegas algorithms**
  - always correct; expected running time (“probably fast”)
  - Examples:
    - randomized Quicksort,
    - randomized algorithm for closest pair
- **Monte Carlo algorithms (mostly correct):**
  - probably correct; guaranteed running time
  - Example: randomized primality test

Quicksort

Unsorted range *A*[*l*, *r*] in array *A*



Quicksort

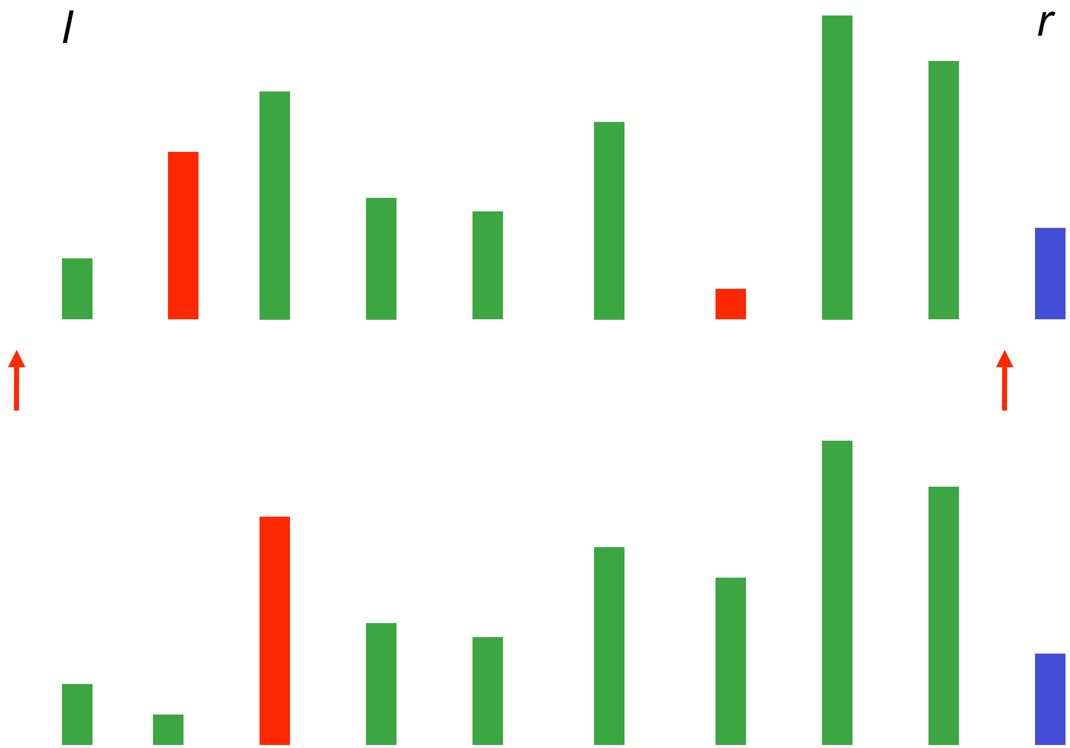
Algorithm: *Quicksort*

Input: **unsorted range [*l*, *r*] in array *A***

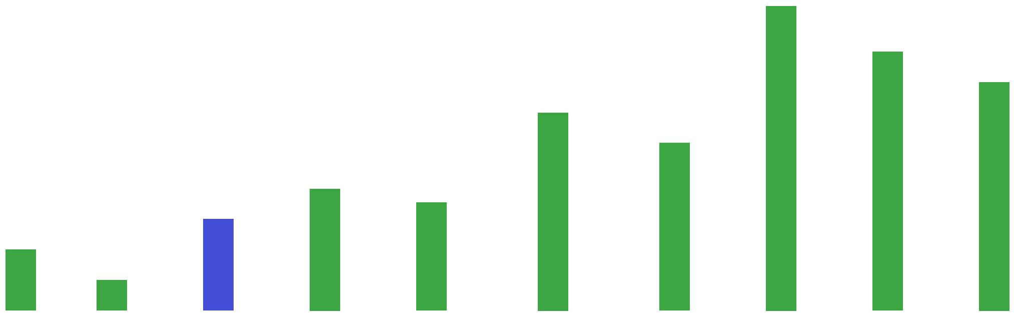
Output: **sorted range [*l*, *r*] in array *A***

```
if r > l
   then choose pivot element p = A[r]
      m = divide(A, l , r)
        /* Divide A according to p:
             A[l],....,A[m – 1] ≤ p ≤ A[m + 1],...,A[r]
         */
      Quicksort(A, l , m - 1)
      Quicksort (A, m + 1, r)
```

The *divide* step



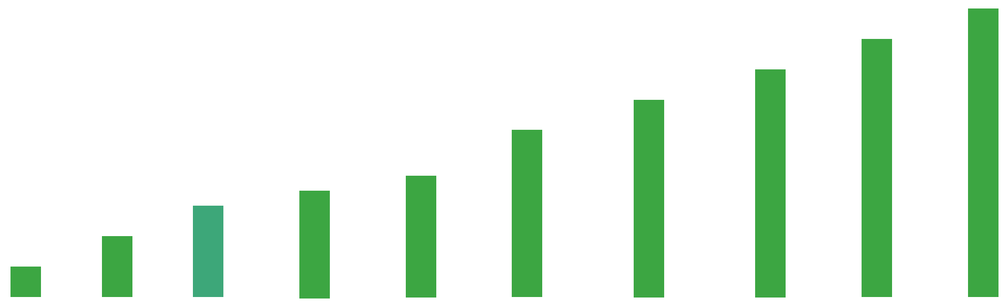
The *divide* step



divide(*A*, *l* , *r*):

- returns the index of the pivot element in *A*
- can be done in time $O(r - l)$

Worst-case input



n elements:

Running time: $(n-1) + (n-2) + \ldots + 2 + 1 = n \cdot (n-1)/2$

Randomized Quicksort

Algorithm: Quicksort

Input: unsorted range [*l*, *r*] in array *A*

Output: sorted range [*l*, *r*] in array *A*

if $r > l$

  then randomly choose a pivot element $p = A[i]$ **in range [*l*, *r*]**

    swap $A[i]$ **and** $A[r]$

    m **= divide(*A*, *l*, *r*)**

    /* Divide *A* according to *p*:

      $A[l],\ldots,A[m-1] \leq p \leq A[m+1],\ldots,A[r]$

    ***/**

    Quicksort(*A*, *l*, *m* - *1*)

    Quicksort(*A*, *m* + *1*, *r*)

Analysis 1

n elements; let S_i be the i-th smallest element

S_1 is chosen as pivot with probability $1/n$:
Sub-problems of sizes 0 and n-1

•
•
•

S_k is chosen as pivot with probability $1/n$:
Sub-problems of sizes k-1 and n-k

•
•
•

S_n is chosen as pivot with probability $1/n$:
Sub-problems of sizes n-1 and 0

Analysis 1

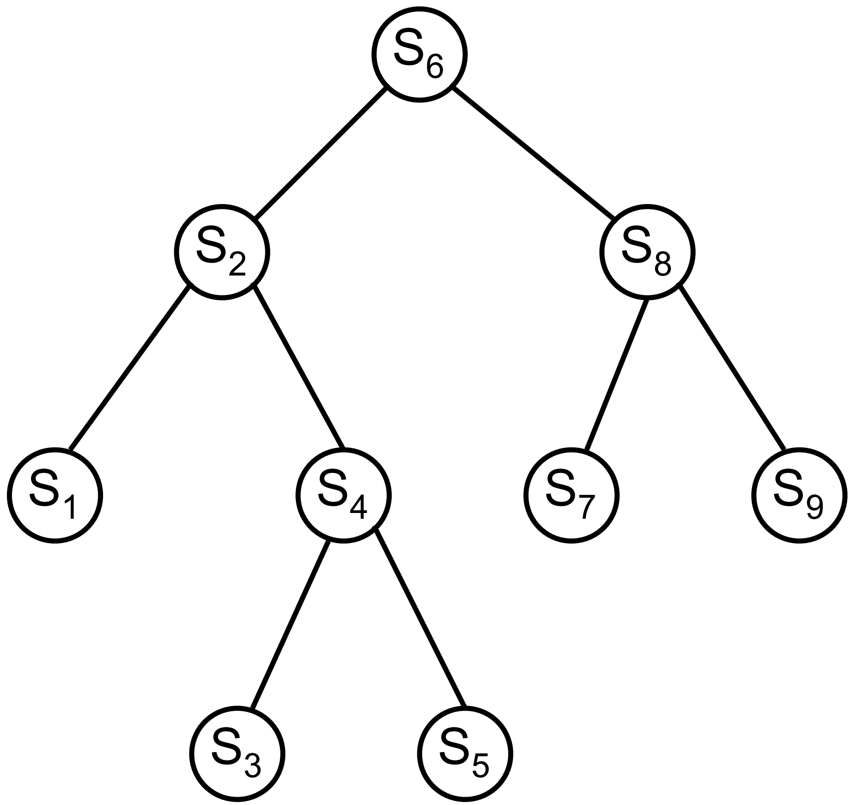
Expected running time:

$$T(n) = \frac{1}{n}\sum_{k=1}^{n}\left(T(k-1) + T(n-k)\right) + \Theta(n)$$

$$= \frac{2}{n}\sum_{k=1}^{n} T(k-1) + \Theta(n)$$

$$= O(n \log n)$$

Analysis 2: Representation of Quicksort as a tree



$\pi = S_6\ S_2\ S_8\ S_1\ S_4\ S_7\ S_9\ S_3\ S_5$

Analysis 2

Expected number of comparisons:

$$X_{ij} = \begin{cases} 1 & \text{if } S_i \text{ is compared with } S_j \\ 0 & \text{otherwise} \end{cases}$$

$$E\left[\sum_{i=1}^{n} \sum_{j>i} X_{ij}\right] = \sum_{i=1}^{n} \sum_{j>i} E[X_{ij}]$$

p_{ij} = probability that S_i is compared with S_j

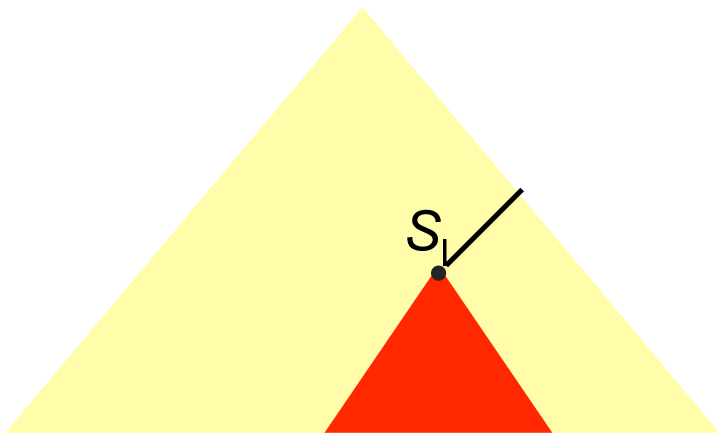
$$E[X_{ij}] = 1 \times p_{ij} + 0 \times (1 - p_{ij}) = p_{ij}$$

Calculation of p_{ij}

- S_i **is compared with** S_j **iff** S_i **or** S_j **is chosen as pivot element in** π **before any other** S_l **,** $i<l<j$**.**

  {S_i **...** S_l **...** S_j**}**

- **Each of the elements** S_i **, ... ,** S_j **is chosen first as the pivot with the same probability.**



$$p_{ij} = 2/(j-i+1)$$

$\{\ldots\ S_i\ \ldots\ S_l\ \ldots\ S_j\ \ldots\}$

Analysis 2

Expected number of comparisons:

$$\sum_{i=1}^{n} \sum_{j>i} p_{ij} = \sum_{i=1}^{n} \sum_{j>i} \frac{2}{j-i+1}$$

$$= \sum_{i=1}^{n} \sum_{k=2}^{n-i+1} \frac{2}{k}$$

$$\leq 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k}$$

$$= 2n \sum_{k=1}^{n} \frac{1}{k}$$

$$H_n = \sum_{k=1}^{n} 1/k \approx \ln n$$

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