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Algorithm Theory

10 Bin Packing

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Problem Definition

- **Given:**
  - n items with sizes $s_1, \ldots , s_n$
  - where $0 < s_i \leq 1$ for $1 \leq i \leq n$.
- **Goal:**
  - Pack items into a minimum number of unit-capacity bins.
- **Example**
  - 7 items with sizes 0.2, 0.5, 0.4, 0.7, 0.1, 0.3, 0.8

Problem Definition

- **Online Bin-Packing:**
  - Items arrive one by one. Each item must be assigned immediately to a bin, without knowledge of any items. Reassignment is not allowed.
- **Offline Bin Packing**
  - All n items are known in advance, i.e. before they have to be packed.

Observations

- **Bin Packing is a combinatorially complex problem**
  - The decision problem of offline Bin Packing is NP-hard
- **At the moment no online polynomial time bounded bin packing algorithm is known that always finds an optimal solution**
  - otherwise P=NP

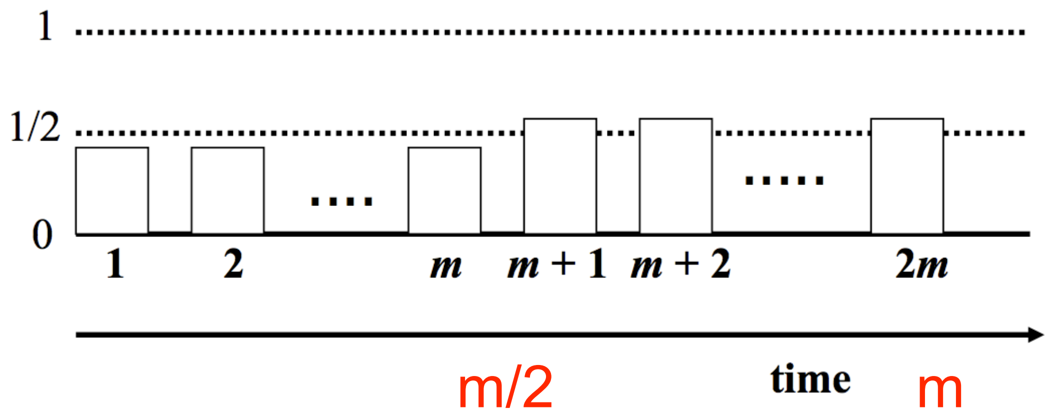
Online Algorithms

Theorem 1

There are inputs that force each online bin packing algorithm to use at least 4/3 OPT bins while OPT is the minimum number of bins necessary.

Proof:

Assumption: online bin packing algorithm *A* always uses less than 4/3 OPT bins



Online Bin Packing

- **After m packets**
  - Optimal solution = m/2 and #bins(A) = b
    by assumption: by $b < 4/3\ m/2 = 2/3\ m$
  - Let $b = b_1 + b_2$ where
    - b_1 = # bins containing one item after round m
    - b_2 = # bins containing two items after round m
  - $m = b_1 + 2 b_2$, i.e. $b_1 = m - 2b_2$
  - Hence: $b = b_1 + b_2 = m - b_2$

Online Bin Packing

- **After 2m packets**
  - Optimal solution = m
  - #bins(A) $\geq$ b + m-b_1 = m+ b_2
  - by assumption: m+b_2 $\leq$ #bins(A) < (4/3) m
    - Hence, b_2 < m/3
  - Since b = m-b_2 (last slide)
    - it follows b = m-b_2 > (2/3) m
    - contradicts b <(2/3) m from last slide

Online Bin Packing

Next-Fit (NF), First-Fit (FF), Best-Fit (BF)

Next-Fit:

Assign an arriving item to the same bin as the preceding item. If it does not fit, open a new bin and place it there.

Theorem 2

(a) For all input sequences I:

$$NF(I) \leq 2\ OPT(I).$$

(b) There exist input sequences I such that:

$$NF(I) \geq 2\ OPT(I) - 2.$$

Next Fit

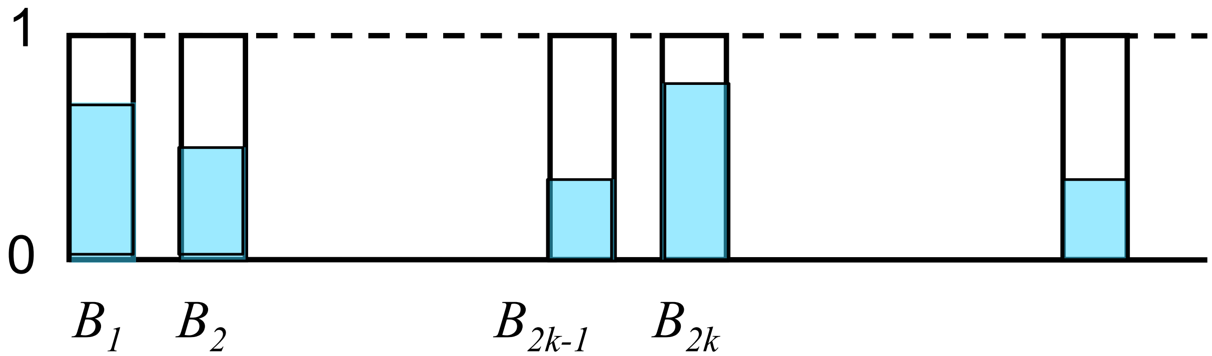
Proof: (a)

Consider two bins B_{2k-1}, B_{2k}, $2k \leq NF(I)$.

The combined amount of the two is larger than 1

$$\sum_{i=1}^{2k} B_i = \sum_{j=1}^{k} B_{2j-1} + B_{2j} > k$$

2k is a lower bound for the optimal solution OPT(I)



Next Fit

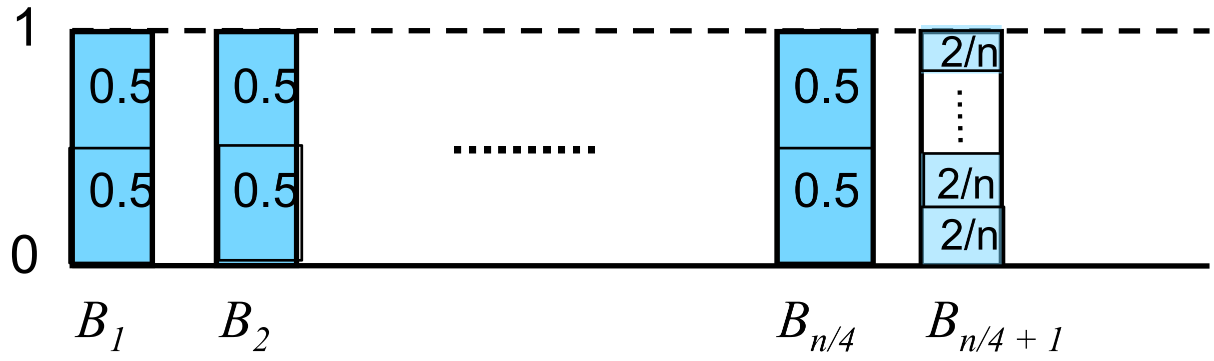
Proof: (b)

Consider an input sequence I of length n

$n \equiv 0 \pmod 4$:

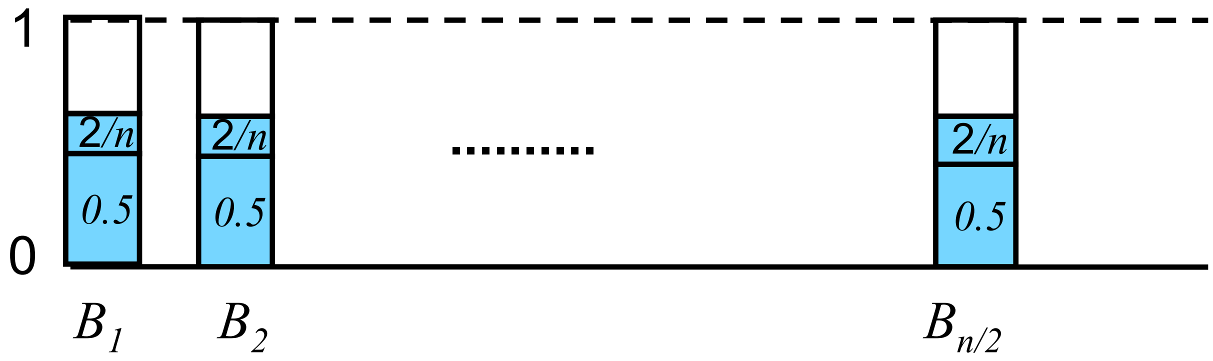
0.5, $2/n$, 0.5, $2/n$, 0.5, ... , 0.5, $2/n$

Optimal packing:



Next Fit

Next Fit yields:



$NF(I)$ = n/2

$OPT(I)$ = n/4+1

First Fit

First Fit:

Assign an arriving item to the first bin (i.e. that was opened earliest) in which it fits. If there is no such bin, open a new one and place it there.

Observation:

At each point in time there is at most one bin that is less than half full

→ $FF(I) \leq 2\,OPT(I)$

First Fit

Theorem 3

(a) For all input sequences I:

$$FF(I) \leq \lceil 17/10\ OPT(I) \rceil$$

(b) There exist input sequences I such that:

$$FF(I) \geq 17/10\ (OPT(I) - 1)$$

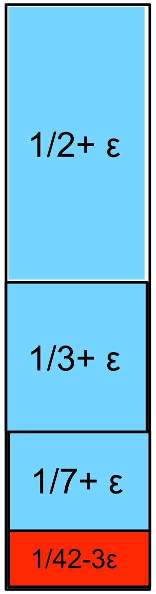
(b´) There exist input sequences I such that:

$$FF(I) = 10/6\ OPT(I)$$

First Fit

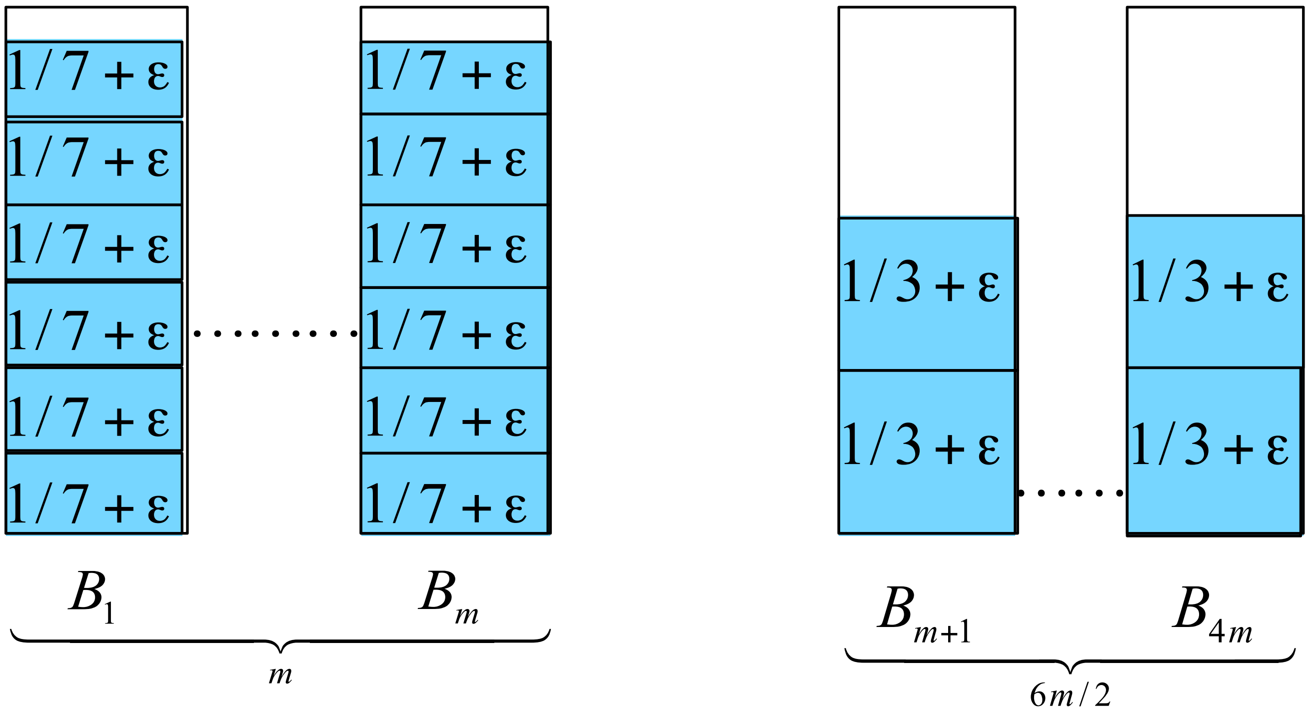
Proof (b`): Input sequence of length $3 \cdot 6m$

$$\underbrace{1/7+\varepsilon,\ldots,1/7+\varepsilon}_{6m},\ \underbrace{1/3+\varepsilon,\ldots,1/3+\varepsilon}_{6m},\ \underbrace{1/2+\varepsilon,\ldots,1/2+\varepsilon}_{6m}$$

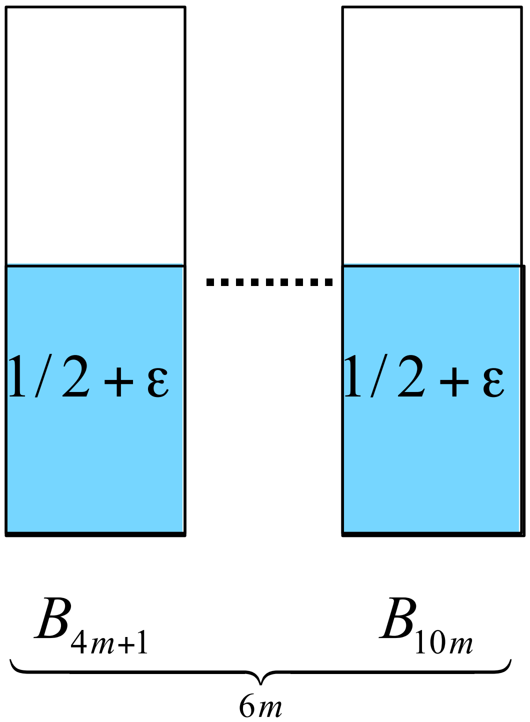


First Fit

First-Fit yields:



First Fit



Best Fit

Best Fit:

Assign an arriving item to the bin in which it fits best (i.e. where it leaves the smallest empty space).

Performance of BF and FF is similar.

Running times on input sequences of length n:

Next Fit:	$O(n)$	
First Fit:	$O(n^2)$	→ O(n log n)
BestFit:	$O(n^2)$	→ O(n log n)

Offline Bin Packing

Prior to the packing, n and $s_1, \ldots, s_n$ are known in advance.

An optimal packing can be found by exhaustive search.

Approach to an offline approximation algorithm:

Initially sort the items in decreasing order of size and assign the larger items first!

First Fit Decreasing (FFD) bzw. **FFNI**

Best Fit Decreasing (BFD)

First Fit Decreasing

Lemma 1

Let I be an input sequence of n objects with sizes

$$s_1 \geq s_2 \geq ..... \geq s_n$$

and let $m = OPT(I)$.

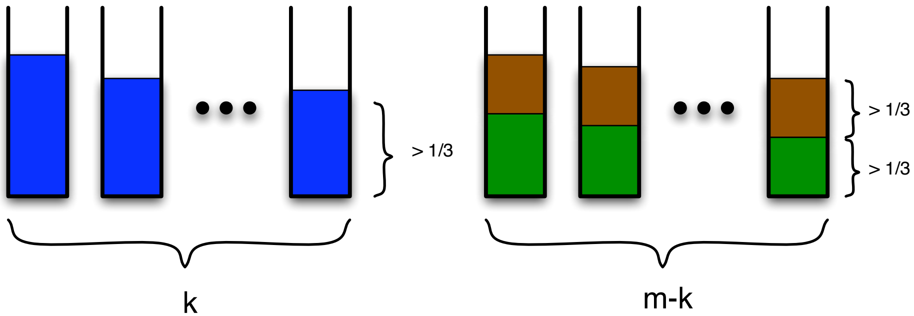
Then, all items placed by FFD into bins

$$B_{m+1}, B_{m+2}, \ldots , B_{FFD(I)}$$

are of size at most 1/3.

Proof of Lemma 1

- **Proof: For $s_1 \geq s_2 \geq ..... \geq s_n$**
  - Let s_i be the first item placed into extra (non-optimal) bin m+1. We show that $s_i \leq 1/3$
  - Assume that $s_i > 1/3$
    - In the configuration of FFD just before s_i is placed we have for all m filled bins:
      - first k bins filled one item
      - next m-k bins with two items



- Optimal solution
  - first k bins are in separate bins.
  - Items in the last m-k bins cannot combined with items of the first k bins.
- If FFD cannot place s_i into the first m bins, then the optimal solution cannot place it it there, too
  - this is a contradiction since m is the optimal number of bins

First Fit Decreasing

Lemma 2

Let I be an input sequence of n objects with sizes

$$s_1 \geq s_2 \geq \ldots\ldots \geq s_n$$

and let $m = OPT(I)$.

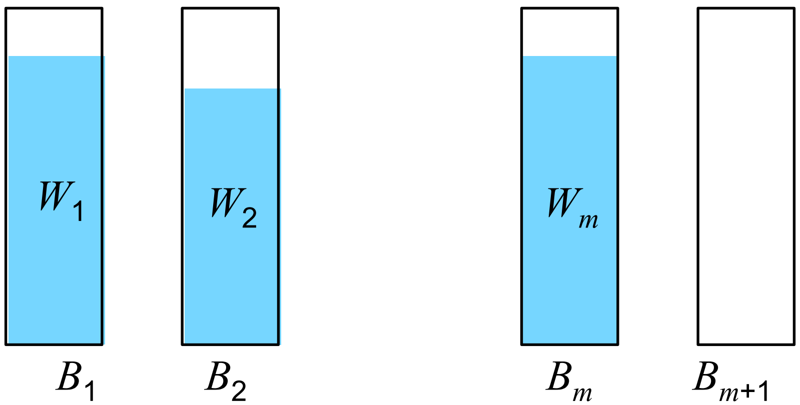
Then, the number of items placed by FFD into bins

$$B_{m+1}, B_{m+2}, \ldots, B_{FFD(I)}$$

is at most $m - 1$.

Proof of Lemma 2

- **Proof:**
  - Assumption:
    - FFD places more than m – 1 items, say $x_1,....,x_m$ into extra bins
    - Then $W_i + x_i > 1$
    - Then the contradiction follows: $$\sum_{i=1}^{m} W_i + x_i > m$$



First Fit Decreasing

Theorem

For all input sequences I:

$$FFD(I) \leq 4/3\ OPT(I) + 1/3$$

Theorem

1. For all input sequences I:

$$FFD(I) \leq 11/9\ OPT(I) + 4.$$

2. There exist input sequences I with:

$$FFD(I) = 11/9\ OPT(I).$$

Proof of First Theorem

- **Proof:**
  - at most m-1 items go into extra bins each have a size of at most 1/3

$$
\begin{aligned}
FFD(I) &= m + \left\lceil \frac{m-1}{3} \right\rceil \\
&\leq m + \frac{m-1}{3} + \frac{2}{3} \\
&= \frac{4}{3}m + \frac{1}{3} \\
&= \frac{4}{3}OPT + \frac{1}{3}
\end{aligned}
$$

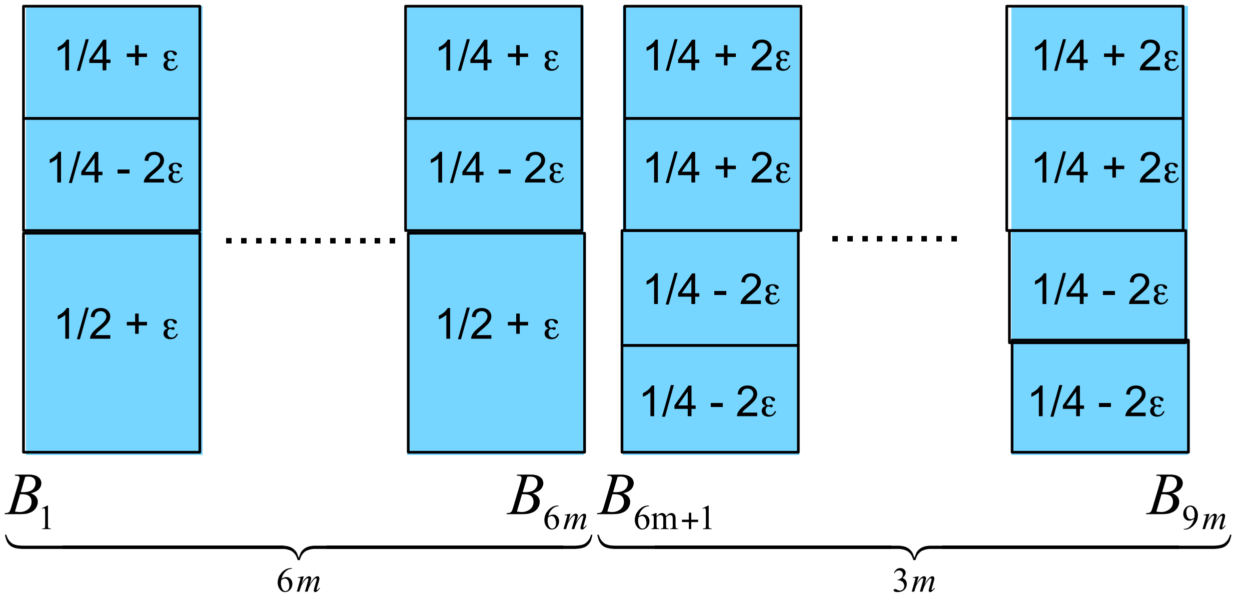
First Fit Decreasing

Proof (b): Input sequence of length $3 \cdot 6m + 12m$

$$\underbrace{1/2+\varepsilon,\ldots,1/2+\varepsilon}_{6m},\underbrace{1/4+2\varepsilon,\ldots,1/4+2\varepsilon}_{6m}$$

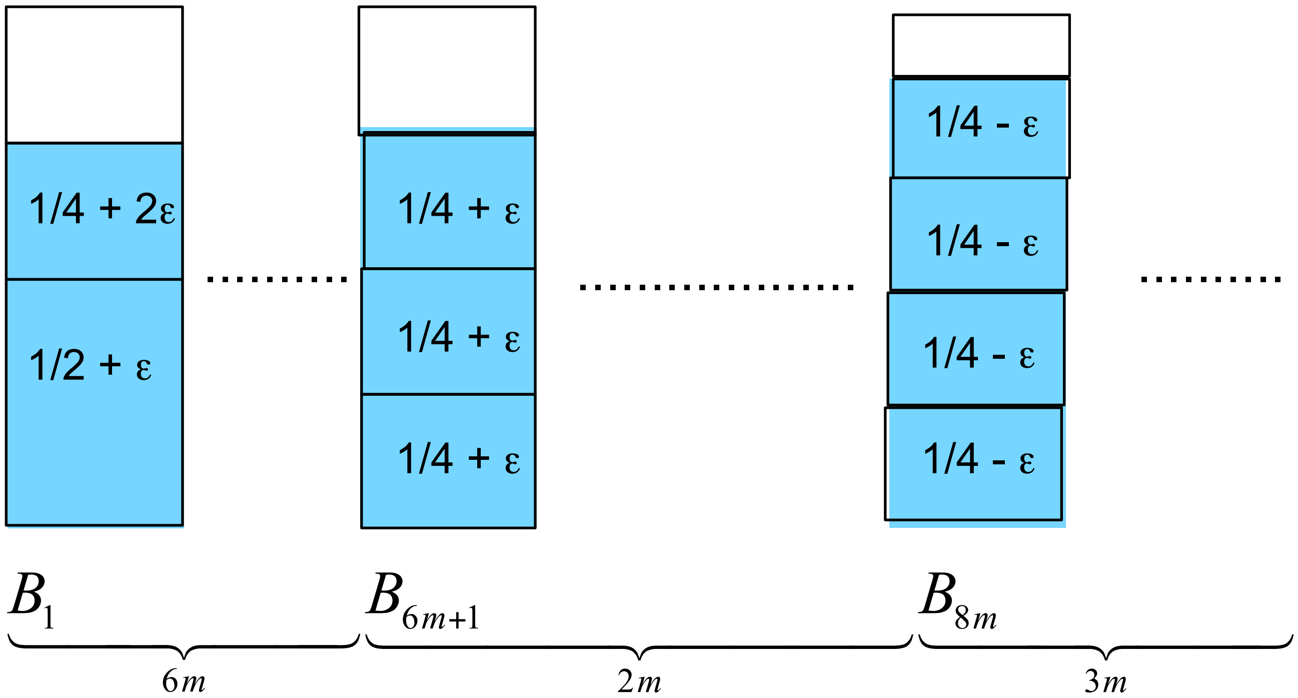
$$\underbrace{1/4+\varepsilon,\ldots,1/4+\varepsilon}_{6m},\underbrace{1/4-2\varepsilon,\ldots,1/4-2\varepsilon}_{12m}$$

Optimal packing:



First Fit Decreasing

First Fit Decreasing yields:



$OPT(I) = 9m$

$FFD(I) = 11m$

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