ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG

Algorithm Theory

18 Minimum Spanning Trees

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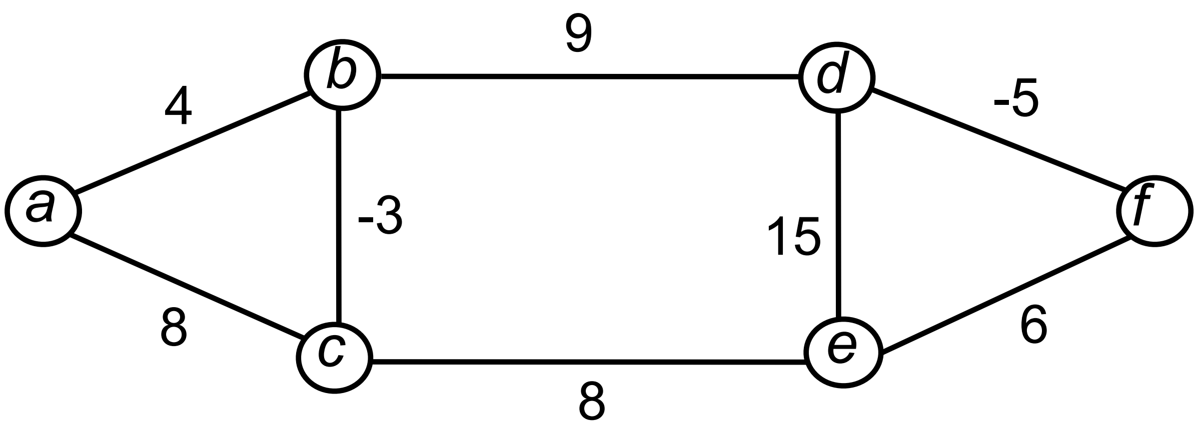
Minimum Spanning Trees

Undirected graph $G = (V, E)$

Weight function $c: E \rightarrow R$

Let $T \subseteq E$ be a tree (connected acyclic subgragh)

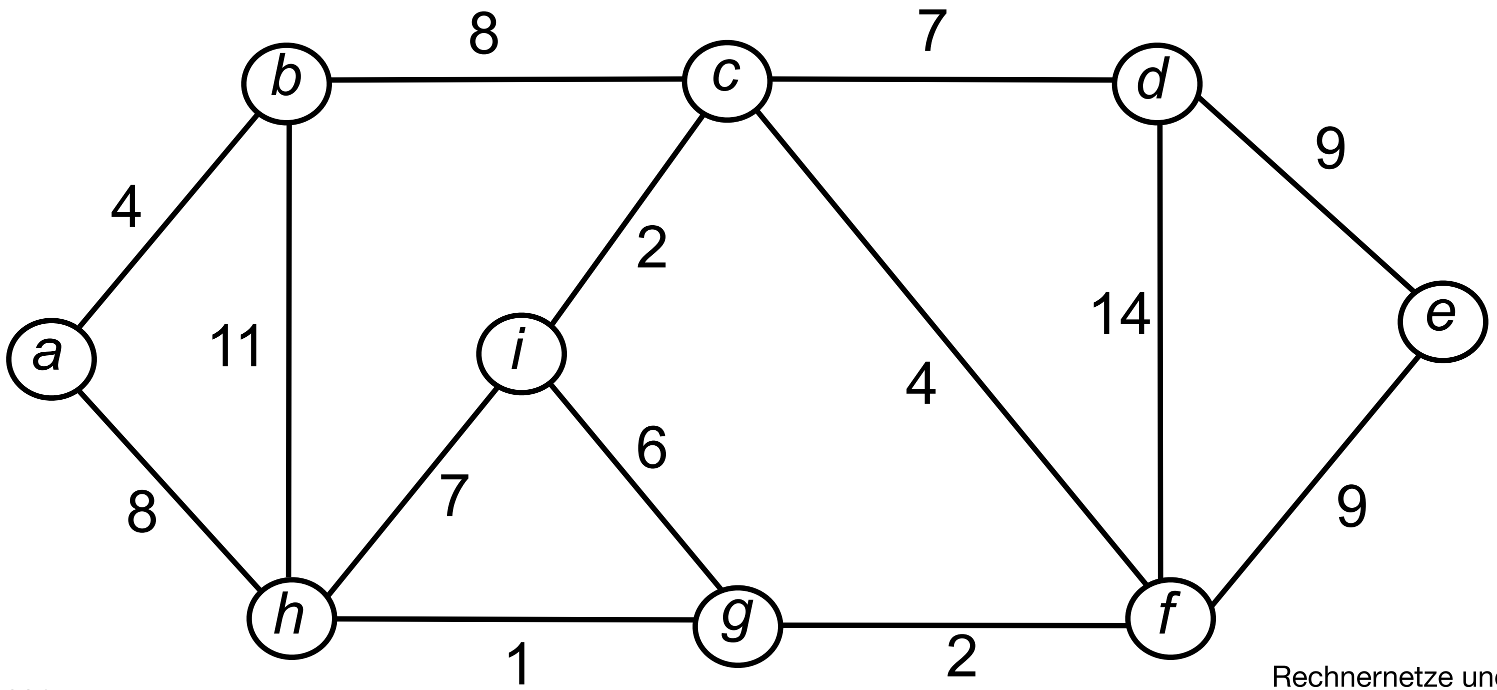
Total weight of T : $$c(T) = \sum_{(u,v)\in T} c(u,v)$$



Minimum Spanning Trees

A tree $T \subseteq E$ connecting all nodes in V with minimum weight is called a

minimum spanning tree



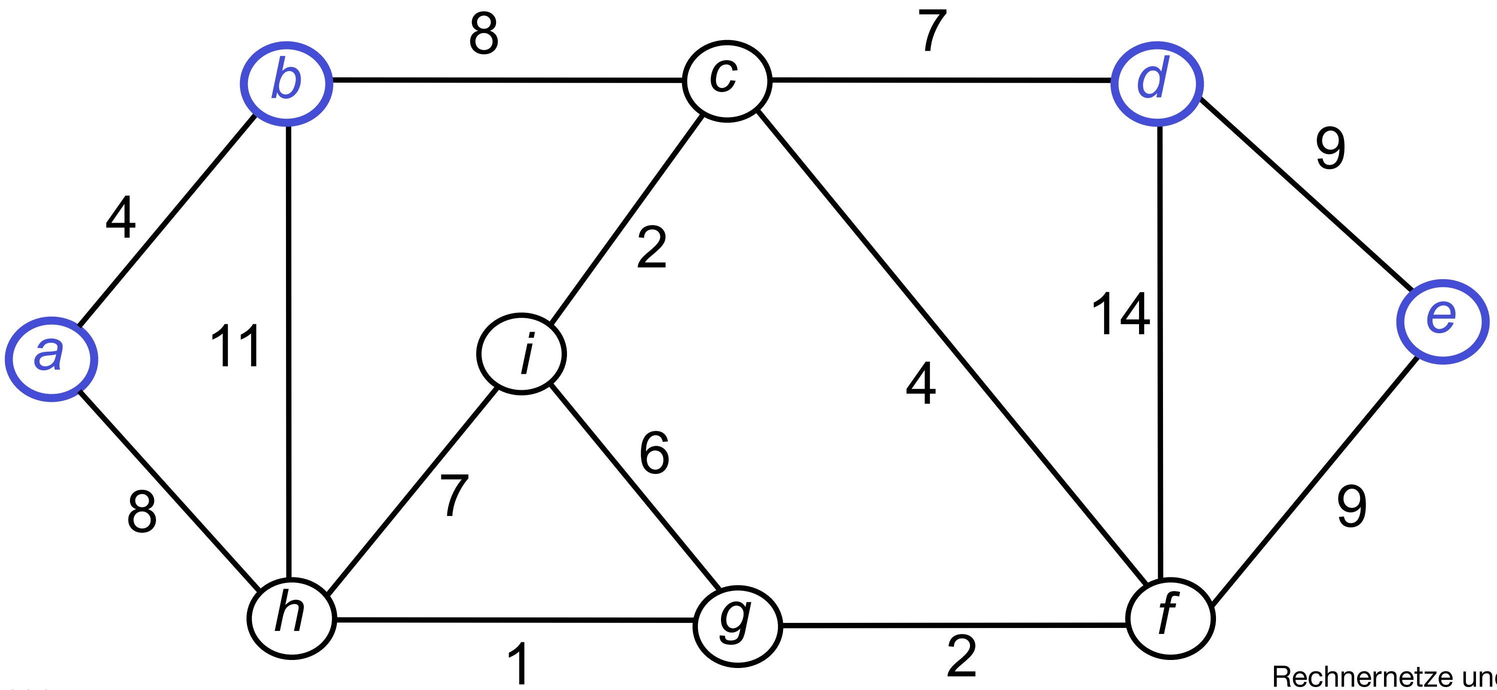
Growing a MST

- **Invariant:**
  - Maintain a set $A \subseteq E$ that is a subste of some minimum spanning tree
- **Definition:**
  - An edge $(u,v) \in E \setminus A$ is a safe edge for A if $A \cup \{(u,v)\}$ is also a subset of some minimum spanning tree

Cuts

A cut $(S, V \setminus S)$ is a partition of V.

An edge (u,v) crosses $(S, V \setminus S)$, if one of its endpoints is in S and the other is in $V \setminus S$.



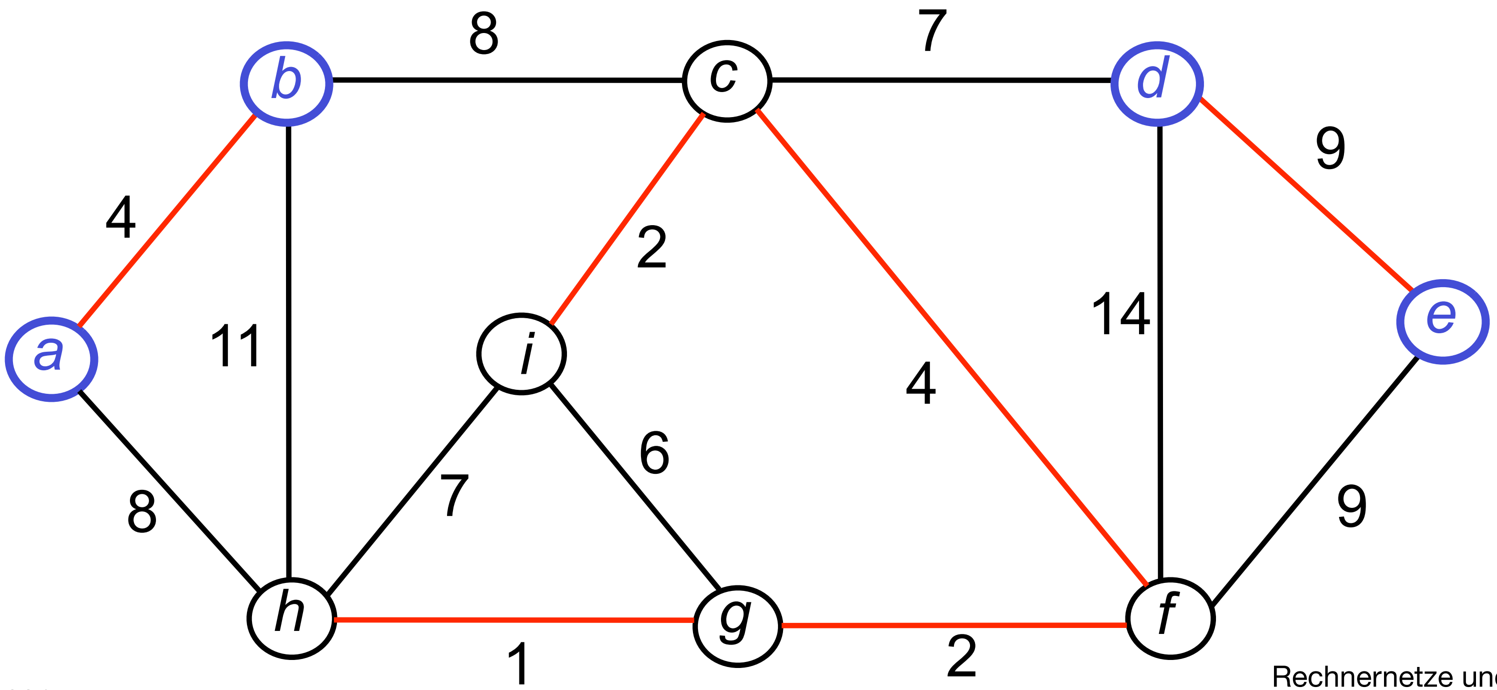
Greedy Approach

Algorithm Generic-MST(G,w)

1. A $\leftarrow \varnothing$
2. **while** A does not form a spanning tree **do**
3. Find an edge (u,v) that is safe for A;
4. A $\leftarrow A \cup \{(u,v)\}$;
5. **endwhile**;

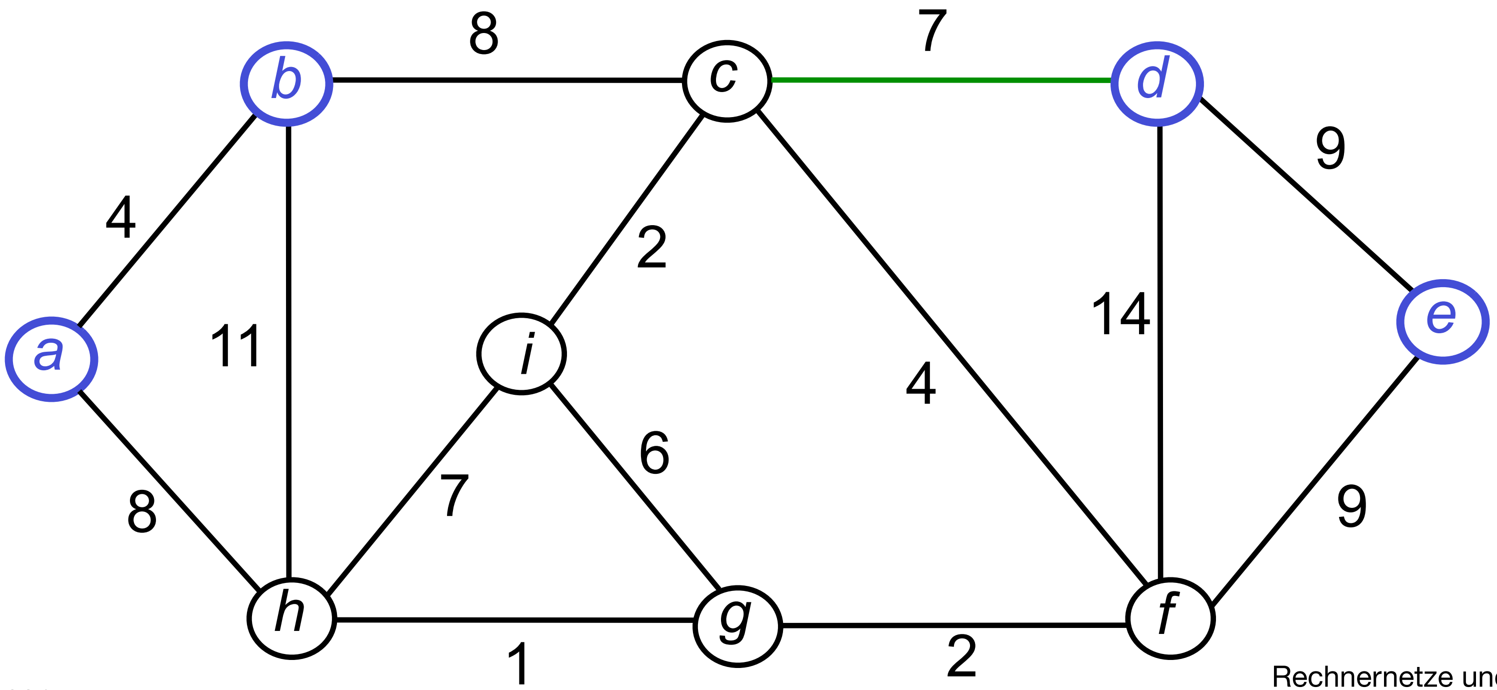
Cuts

- A cut respects a set A of edges if no edges in A crosses the cut



Cuts

An edge is a light edge crossing a certain cut if its weight is the minimum of any edge crossing the cut



Safe Edges

Theorem:

Let A be a subset of some minimum spanning Tree T, and let (S, V\S) be a cut that respects A, If (u,v) is a light edge crossing (S, V\S) then (u,v) is safe for A.

Proof

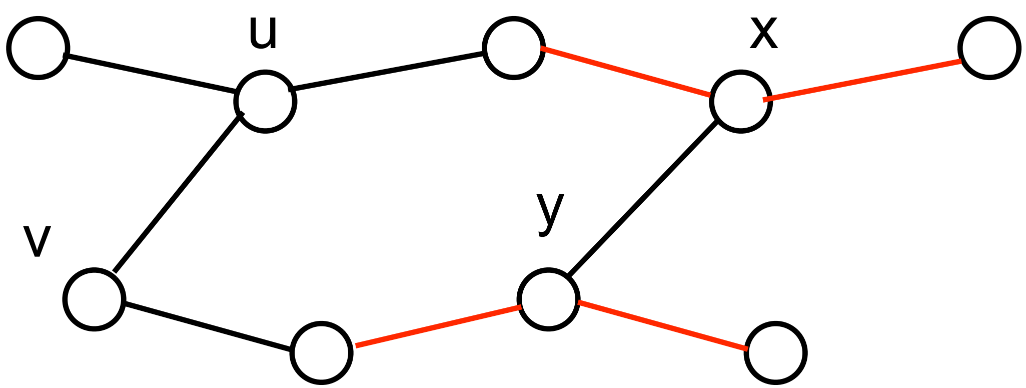
Case 1: $(u,v) \in T$: o.k.

Case 2: $(u,v) \notin T$:

We construct another minimum spanning tree T' with $(u,v) \in T$ and $A \subseteq T'$.

Safe Edges

- **Adding (u,v) to T yields a cycle.**
  - On this cycle, there is at least one edge (x,y) in T that also crosses the cut



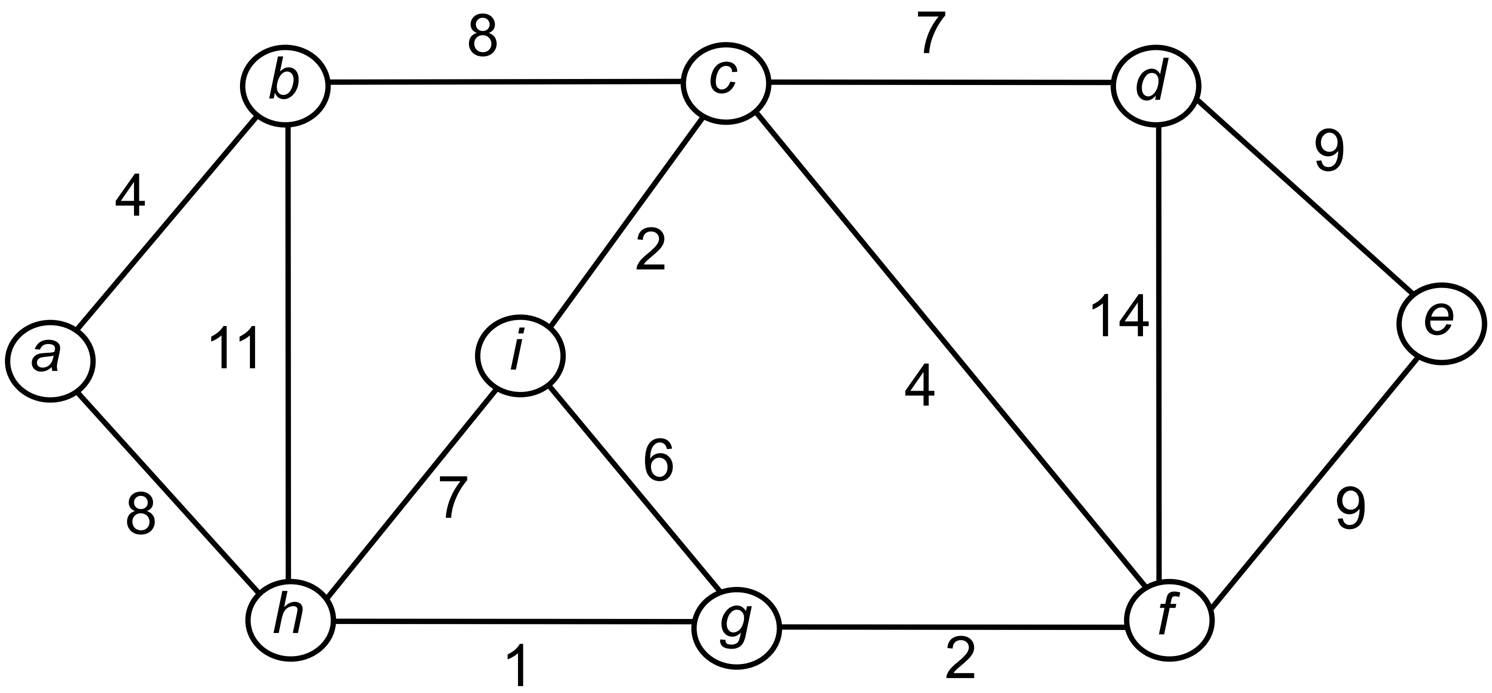
- T‘ = T \ {(x,v)} ∪ {(u,v)} is a minimum spanning tree, since
  - $w(T') = w(T) - w(x,y) + w(u,v) \leq w(T)$

The Graph G_A

- **$G_A=(V,A)$**
  - is a forest, i.e. a collection of trees
  - at the beginning, when $A = \emptyset$, each tree consists of a single vertex
  - any safe edge for A connects distinct trees
- **Corollary**
  - Let B be a tree in $G_A=(V,A)$, If (u,v) is a light edge connecting B to some other tree in G_A, then (u,v) is safe for A.
- **Proof:**
  - (B,V\B) respsects A and (u,v) is a light edge for this cut

Kruskal's Algorithm

Always choose an edge of smallest weight that connects two trees B_1 and B_2 in G_A



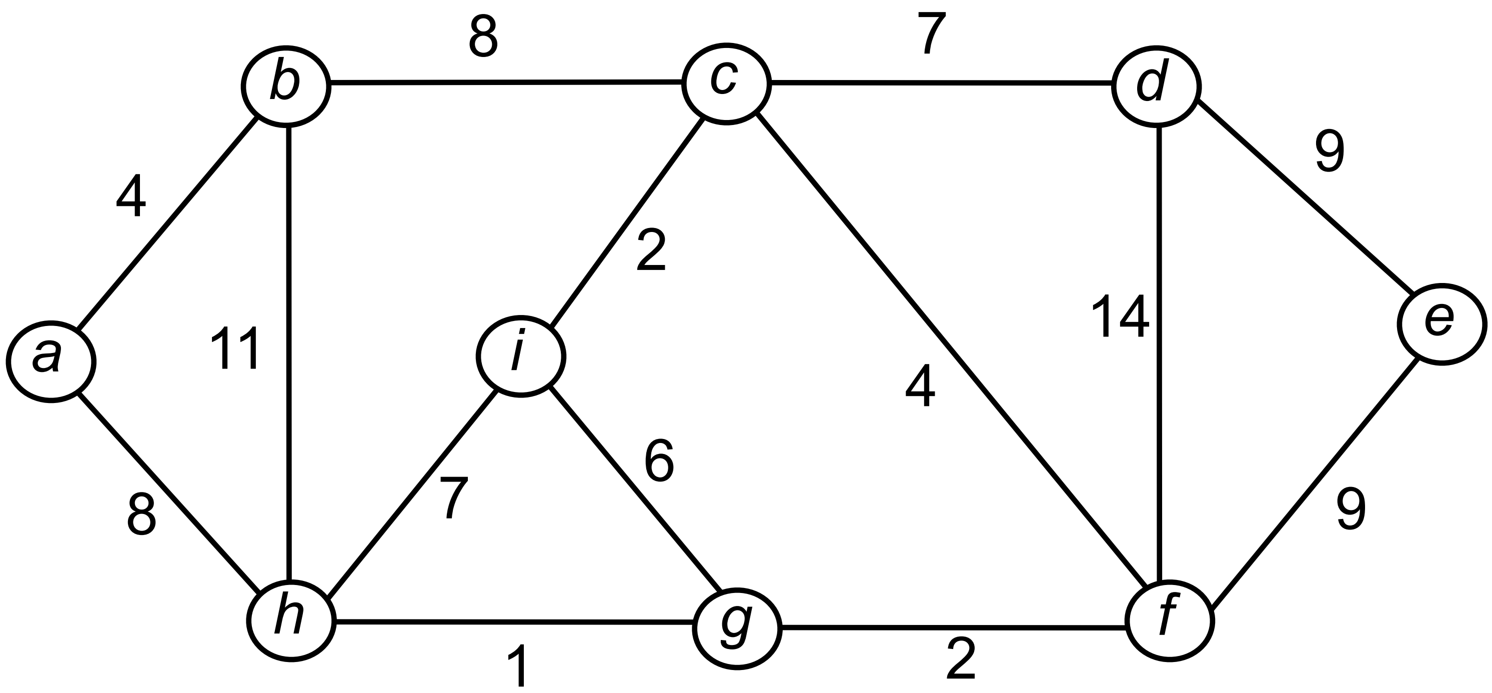
Algorithm of Kruskal

1. $A \leftarrow \emptyset$;
2. **for all** $v \in V$ **do** $B_v \leftarrow \{ v \}$; **endfor;**
3. Generate a list L of all edges in E, sorted in non-decreasing order of weight
4. **for all** (u,v) in L **do**
5. $B_1 \leftarrow$ FIND(u); $B_2 \leftarrow$ FIND(v);
6. **if** $B_1 \neq B_2$ **then**
7. $A \leftarrow A \cup \{(u,v)\}$; UNION ($B_1$, B_2, B_1);
8. **endif;**
9. **endfor;**

Running time: $O(m\,\alpha(m,n) + m + m \log m) = O(m \log m)$

Prim‘s Algorithm

A is always a single tree.
Start from an arbitrary root vertex r.
In each step, add a light edge to A that connects A to a vertex in V \ A



Implementation

Q : Priority Queue containing all vertices $v \in V \setminus A$.

key of vertex v: minimum weight of any edge connecting v to a vertex in A

For a node v, let $p[v]$ denote the parent of v in the tree.

$A = \{(v, p[v]) : v \in V \setminus (\{r\} \cup Q)\}$

r = root vertex

Prim‘s Algorithm

```
for all v ∈ V do Insert(Q, ∞, v); endfor;
Choose a root vertex w ∈ V;
DecreaseKey(Q, 0, w);  p[w] ← nil;
while ¬Empty(Q) do
    (d, u) ← DeleteMin(Q);
    for all (u,v) ∈ E do
        if v ∈ Q and w(u,v) < key of v then
            DecreaseKey(Q, w(u,v), v);   p[v] ← u;
        endif;
    endfor;
endwhile;
```

Running time: **O(** n **log** $n + m$ **)**

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