Peer-to-Peer Networks

6. Analysis of DHT

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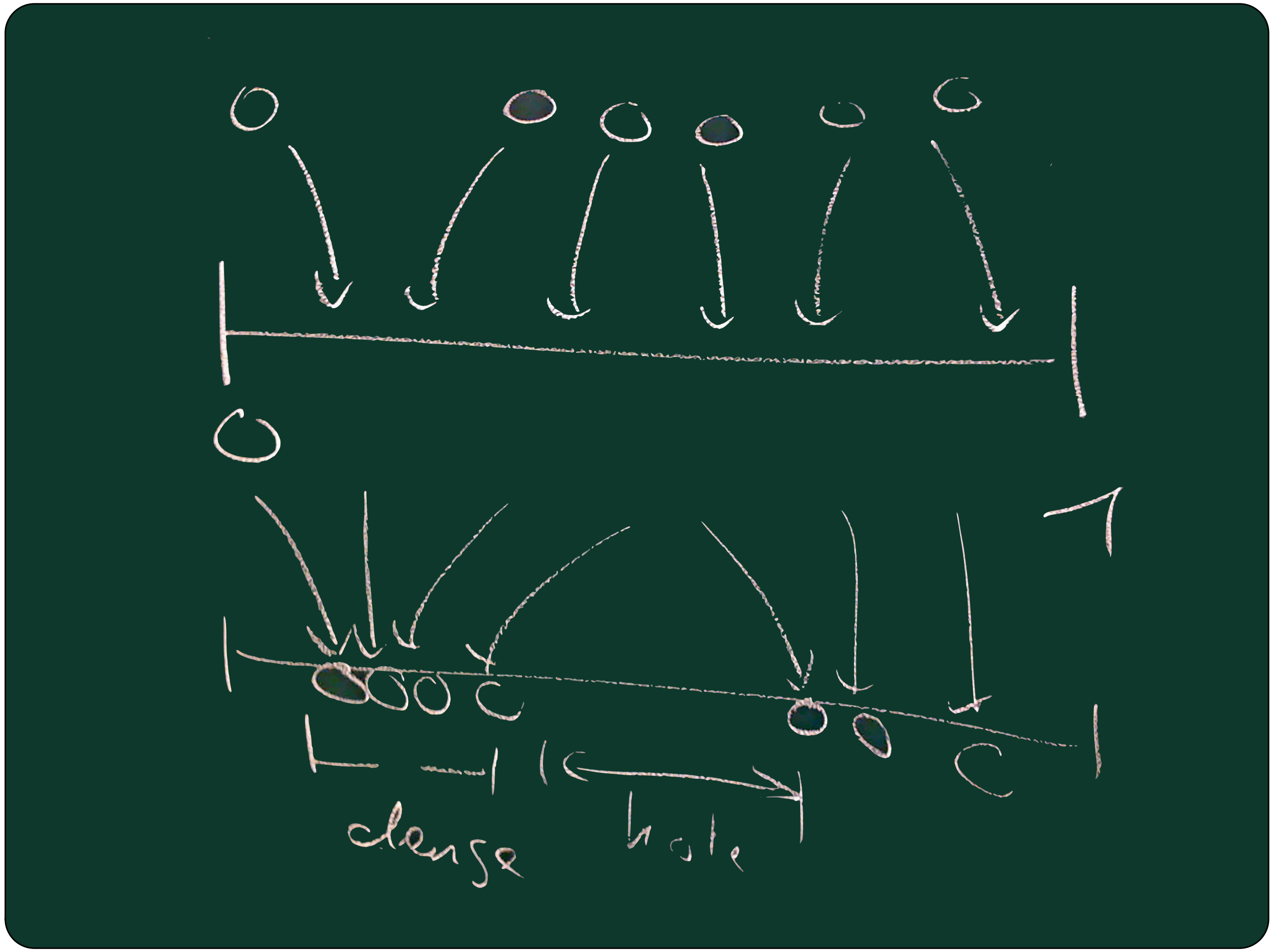
Holes and Dense Areas



Size of Holes

- Theorem
  - If n elements are randomly inserted into an array [0,1[then with constant probability there is a „hole“ of size Ω(log n/n), i.e. an interval without elements.
- Proof
  - Consider an interval of size log n / (4n)
  - The chance not to hit such an interval is (1-log n/(4n))
  - The chance that n elements do not hit this interval is

$$\left(1-\frac{\log n}{4n}\right)^{n} = \left(1-\frac{\log n}{4n}\right)^{\frac{4n}{\log n}\frac{\log n}{4}} \geq \left(\frac{1}{4}\right)^{\frac{1}{4}\log n} = \frac{1}{\sqrt{n}}$$

  - The expected number of such intervals is more than 1.
  - Hence the probability for such an interval is at least constant.

Proof of Dense Areas

$$\left(\frac{1}{4}\right)^{\frac{1}{4}\cdot\log n} = 2^{\left(\frac{1}{4}\log n\right)\overbrace{\log_2 \frac{1}{4}}^{-2}}$$

$$= 2^{\left(-\frac{1}{2}\right)\cdot\log n}$$

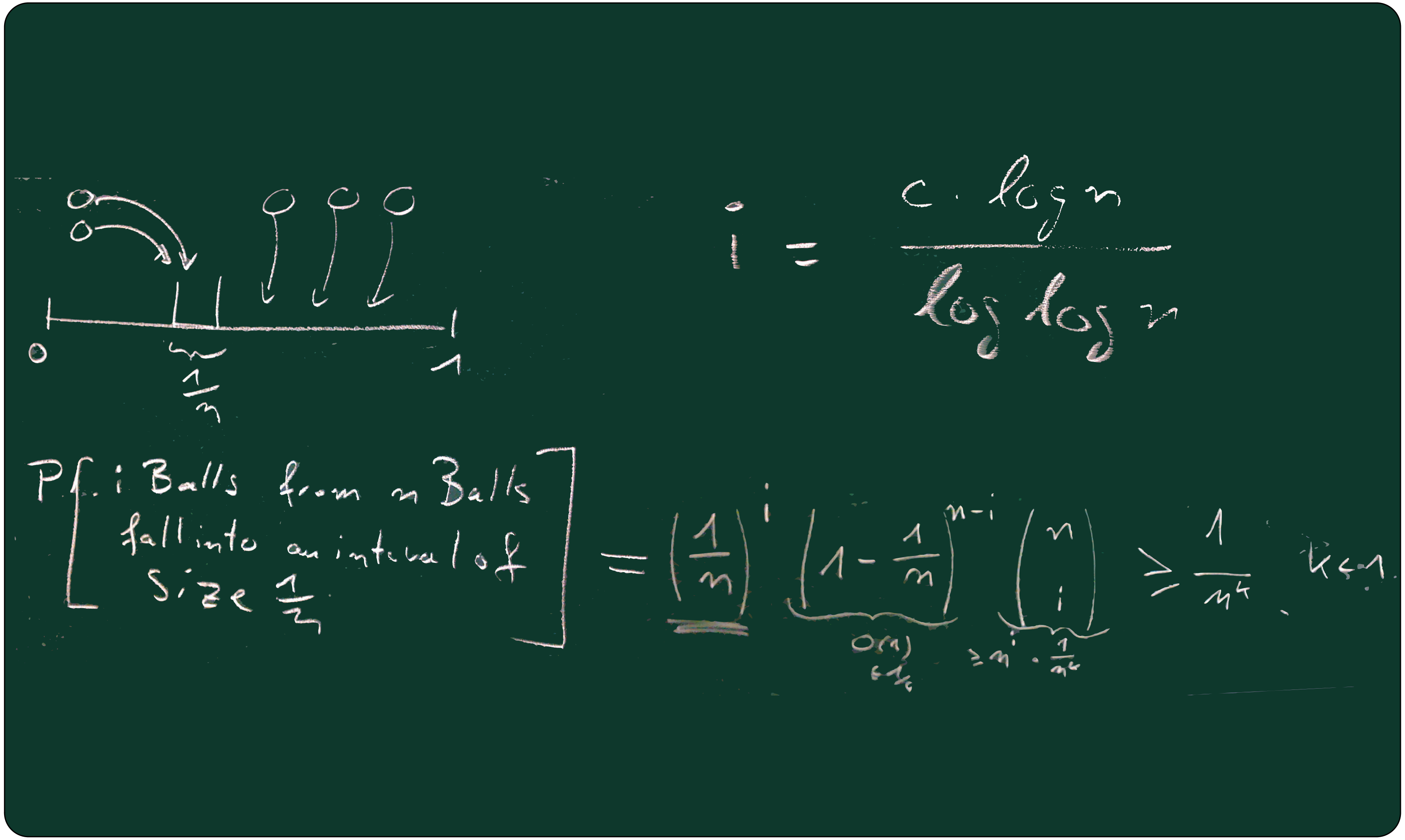
$$= n^{-\frac{1}{2}} = \frac{1}{\sqrt{n}}$$

Expectation: $\frac{4n}{\log n}\cdot\frac{1}{\sqrt{n}} = \frac{4\sqrt{n}}{\log n}$

Dense Spots

- Theorem
  - If n elements are randomly inserted into an array [0,1[then with constant probability there is a dense interval of length 1/n with at least Ω(log n/ (log log n)) elements.
- Proof
  - The probability to place exactly i elements in to such an interval is $$\left(\frac{1}{n}\right)^i \left(1-\frac{1}{n}\right)^{n-i} \binom{n}{i}$$
  - for i = c log n / (log log n) this probability is at least 1/ n^k for an appropriately chosen c and k<1
  - Then the expected number of intervals is at least 1

Proof of Dense Areas



$$i = \frac{c \cdot \log n}{\log \log n}$$

$$P\left[\begin{array}{l}i \text{ Balls from } n \text{ Balls} \\ \text{fall into an interval of} \\ \text{Size } \frac{1}{n}\end{array}\right] = \left(\frac{1}{n}\right)^i \underbrace{\left(1 - \frac{1}{n}\right)^{n-i}}_{\substack{O(1) \\ \leq 1/e}} \underbrace{\binom{n}{i}}_{\geq n^i \cdot \frac{1}{n^k}} \geq \frac{1}{n^k} \quad k < 1.$$

Proof of Dense Areas

$$\frac{1}{4} \underset{m \geq 2}{\leq} \left(1 - \frac{1}{m}\right)^{m} \leq \frac{1}{e}$$

$$\left(1 - \frac{1}{n}\right)^{n-i} = \left(1 - \frac{1}{n}\right)^{n \frac{n-i}{n}}$$

$$\geq \left(\frac{1}{4}\right)^{1 - \frac{i}{n}}$$

$$\geq \frac{1}{4}$$

Proof of Dense Areas

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-i+1)}{i!}$$

$$\geq \frac{\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{n-i+1}{n}}{i^i}\; n^i \qquad \frac{i}{n} \leq \frac{1}{2}$$

$$\geq \underbrace{\left(1-\frac{i-1}{n}\right)^{n-i}}_{\left(1-\frac{i-1}{n}\right)^{\frac{n}{i-1}\cdot\frac{n-i}{n}(i-1)}} \cdot \frac{n^i}{i^i}$$

$$\geq \left(\frac{1}{4}\right)^{\left(1-\frac{i}{n}\right)(i-1)} \geq \left(\frac{1}{4}\right)^{\frac{1}{2}i} = \left(\frac{1}{2}\right)^i$$

Proof of Dense Areas

$$\left(\frac{1}{2}\right)^i \cdot \frac{1}{i^i} = 2^{-i-i\cdot\ln i} \geq \underbrace{2^{-k\cdot\log n}}_{\frac{1}{n^k}}$$

$$\underset{i(1+\ln i)}{i + i\cdot\ln i} \leq \frac{c\cdot\log n}{\log\log n}\left(1+\ln c+\ln\log n - \ln\log\log n\right)$$

$$\leq \frac{c\cdot\log n}{\underset{1}{\log\log n}}\left(1+\ln c+(\ln 2)\right)\overset{\checkmark}{\log}\log n$$

$$= \overbrace{c(1+\ln c+\ln 2)}^{k}\cdot\log n$$

Averaging Effect

- Theorem
  - If $\Theta(n \log n)$ elements are randomly inserted into an array [0,1[then with high probability in every interval of length 1/n there are $\Theta(\log n)$ elements.

Excursion

- Markov-Inequality
  - For random variable X>0 with E[X] > 0:

$$\mathbf{P}[X \geq k \cdot \mathbf{E}[X]] \leq \frac{1}{k}$$

- Chebyshev

$$\mathbf{P}[|X - \mathbf{E}[X]| \geq k] \leq \frac{\mathbf{V}[X]}{k^2}$$

  - for Variance

$$\mathbf{V}[X] = \mathbf{E}[X^2] - \mathbf{E}[X]^2$$

- Stronger bound: Chernoff

Chernoff-Bound

- Theorem Chernoff Bound
  - Let $x_1,\ldots,x_n$ independent Bernoulli experiments with
    - $P[x_i = 1] = p$
    - $P[x_i = 0] = 1-p$
  - Let

$$S_n = \sum_{i=1}^{n} x_i$$

  - Then for all c>0

$$\mathbf{P}[S_n \geq (1+c)\cdot\mathbf{E}[S_n]] \quad \leq \quad e^{-\frac{1}{3}\min\{c,c^2\}pn}$$

  - For 0≤c≤1

$$\mathbf{P}[S_n \leq (1-c)\cdot\mathbf{E}[S_n]] \quad \leq \quad e^{-\frac{1}{2}c^2 pn}$$

Proof of 1st Chernoff Bound

- We show

$$\boxed{\mathrm{P}[S_n \ \geq \ (1+c)\mathbf{E}[S_n]] \ \leq \ e^{-\frac{\min\{c,c^2\}}{3}pn}}$$

- Für t>0:

$$\mathrm{P}[S_n \geq (1+c)pn] \ = \ \mathrm{P}[e^{tS_n} \geq e^{t(1+c)pn}]$$

$$\boxed{\frac{1}{k} \ \leq \ e^{-\frac{\min\{c,c^2\}}{3}pn}}$$

$$k = e^{t(1+c)pn}/E[e^{t \cdot S_n}]$$

- Markov yields:

$$\mathbf{P}\left[e^{tS_n} \geq k\mathbf{E}\left[e^{tS_n}\right]\right] \ \leq \ \frac{1}{k}$$

- To do: Choose t appropriately

Proof of 1st Chernoff Bound

- We show $\boxed{\frac{1}{k} \leq e^{-\frac{\min\{c,c^2\}}{3}pn}}$
- where $k = e^{t(1+c)pn} / E[e^{t \cdot S_n}]$

Independence of random variables x_i

$$\begin{aligned}
\mathbf{E}[e^{tS_n}] &= \mathbf{E}\left[e^{t\sum_{i=1}^{n} x_i}\right] \\
&= \mathbf{E}\left[\prod_{i=1}^{n} e^{tx_i}\right] \\
&= \prod_{i=1}^{n} \mathbf{E}\left[e^{tx_i}\right] \\
&= \prod_{i=1}^{n} (e^0(1-p) + e^t p) \\
&= (1-p+e^t p)^n \\
&= (1+(e^t-1)p)^n
\end{aligned}$$

- Next we show:

$$e^{-t(1+c)pn} \cdot (1+p(e^t-1))^n \leq e^{-\frac{\min\{c,c^2\}}{3}pn}$$

Proof of 1st Chernoff Bound

Show:

$$\boxed{e^{-t(1+c)pn} \cdot (1+p(e^t-1))^n \le \ e^{-\frac{\min\{c,c^2\}}{3}pn}}$$

where: $t = \ln(1+c) > 0$

$$\begin{aligned} e^{-t(1+c)pn} \cdot (1+p(e^t-1))^n &\le e^{-t(1+c)pn} \cdot e^{pn(e^t-1)} \\ &= e^{-t(1+c)pn+pn(e^t-1)} \\ &= e^{-(1+c)\ln(1+c)pn+cpn} \\ &= e^{(c-(1+c)\ln(1+c))pn} \end{aligned}$$

Next to show

$$\boxed{(1+c)\ln(1+c) \ge c + \tfrac{1}{3}\min\{c,c^2\}}$$

Proof of 1st Chernoff Bound

To show for c>1:

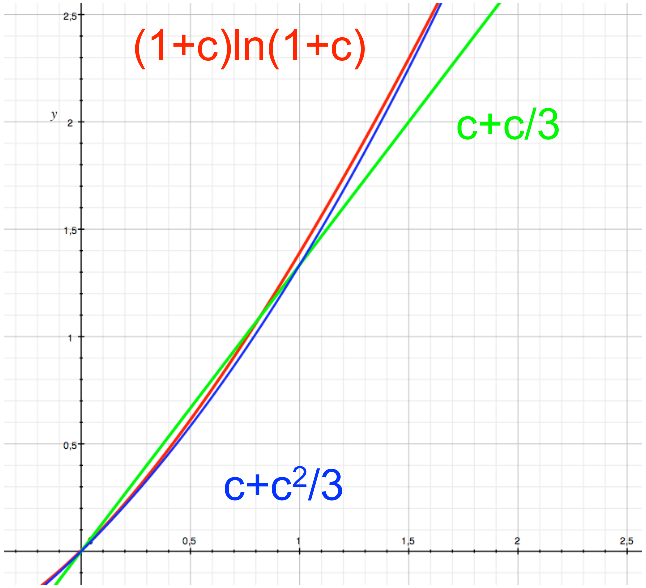
$$(1+c)\ln(1+c) \geq c + \frac{1}{3}c$$

For c=1: 2 ln(2) > 4/3

Derivative:

- left side: ln(1+c)
- right side: 4/3

- For c>1 the left side is larger than the right side since
  - ln(1+c)>ln (2) > 4/3
- Hence the inequality is true for c>0.



Proof of 1st Chernoff Bound

To show for c< 1:

$$\boxed{(1+c)\ln(1+c) \geq c + \frac{1}{3}c^2}$$

For x>0:

$$\frac{d\ln(1+x)}{dx} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots$$

Hence

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$$

By multiplication

$$(1+x)\ln(1+x) = x + \left(1 - \frac{1}{2}\right)x^2 - \left(\frac{1}{2} - \frac{1}{3}\right)x^3 + \left(\frac{1}{3} - \frac{1}{4}\right)x^4 - \ldots$$

Substitute (1+c) ln(1+c) which gives for c∈(0,1):

$$(1+c)\ln(1+c) \geq c + \frac{1}{2}c^2 - \frac{1}{6}c^3 \geq c + \frac{1}{3}c^2$$

Chernoff-Bound

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  - Let $x_1,...,x_n$ independent Bernoulli experiments with
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    - $P[x_i = 0] = 1\text{-}p$
  - Let

$$S_n = \sum_{i=1}^{n} x_i$$

  - Then for all c>0

$$\mathbf{P}[S_n \geq (1+c) \cdot \mathbf{E}[S_n]] \quad \leq \quad e^{-\frac{1}{3}\min\{c,c^2\}pn}$$

  - For 0≤c≤1

$$\mathbf{P}[S_n \leq (1-c) \cdot \mathbf{E}[S_n]] \quad \leq \quad e^{-\frac{1}{2}c^2pn}$$

Proof of 2nd Chernoff Bound

- **We show** $\mathrm{P}[S_n \leq (1-c)\mathbf{E}[S_n]] \;\leq\; e^{-\frac{c^2}{2}pn}$.
- **For t<0:** $\mathrm{P}[S_n \leq (1-c)pn] \;=\; \mathrm{P}[e^{tS_n} \geq e^{t(1-c)pn}]$

$$\boxed{\frac{1}{k} \leq \; e^{-\frac{c^2}{2}pn}}$$

$$k = e^{t(1-c)pn}/\mathbf{E}[e^{t\cdot S_n}]$$

- **Markov yields:** $\mathrm{P}\left[e^{tS_n} \geq k\mathbf{E}\left[e^{tS_n}\right]\right] \;\leq\; \frac{1}{k}$
- **To do: Choose t appropriately**

Proof of 2nd Chernoff Bound

- **We show** $$\boxed{\frac{1}{k} \leq e^{-\frac{c^2}{2}pn}}$$

- **where** $$k = e^{t(1-c)pn} / \mathbf{E}[e^{t \cdot S_n}]$$

Independence of random variables x_i →

$$\boxed{\begin{aligned}
\mathbf{E}[e^{tS_n}] &= \mathbf{E}\left[e^{t\sum_{i=1}^{n} x_i}\right] \\
&= \mathbf{E}\left[\prod_{i=1}^{n} e^{tx_i}\right] \\
&= \prod_{i=1}^{n} \mathbf{E}\left[e^{tx_i}\right] \\
&= \prod_{i=1}^{n} (e^0(1-p) + e^t p) \\
&= (1 - p + e^t p)^n \\
&= (1 + (e^t - 1)p)^n
\end{aligned}}$$

- **Next we show:**

$$\boxed{e^{-t(1-c)pn} \cdot (1 + p(e^t - 1))^n \leq e^{-\frac{c^2}{2}pn}}$$

Proof of 2nd Chernoff Bound

We show $$\boxed{e^{-t(1-c)pn} \cdot (1+p(e^t-1))^n \;\le\; e^{-\frac{c^2}{2}pn}}$$

where: $$t = \ln(1-c)$$

$$\begin{aligned} e^{-t(1-c)pn} \cdot (1+p(e^t-1))^n &\le e^{-t(1-c)pn} \cdot e^{pn(e^t-1)} \\ &= e^{-t(1-c)pn+pn(e^t-1)} \\ &= e^{-(1-c)\ln(1-c)pn-cpn} \end{aligned}$$

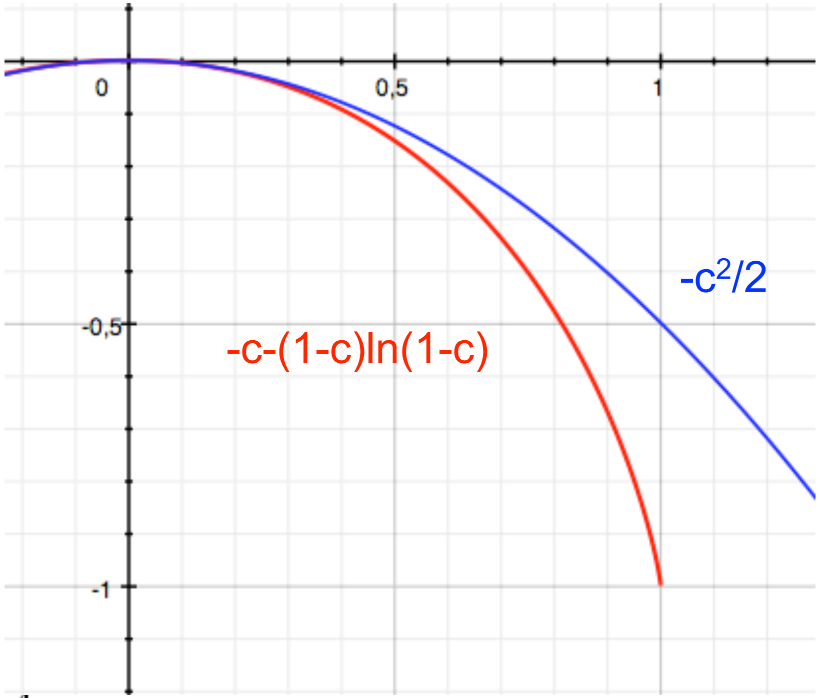
$1+x \le e^x$

Next to show $$\boxed{-c-(1-c)\ln(1-c) \le -\frac{1}{2}c^2}$$

Proof of 2nd Chernoff Bound

To prove:

$$-c-(1-c)\ln(1-c) \le -\frac{1}{2}c^2$$



For c=0 we have equality

Derivative of left side: ln(1-c)

Derivative of right side: -c

Now

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$$

This implies

$$\ln(1-c) = -c - \frac{1}{2}c^2 - \frac{1}{3}c^3 - \ldots < -c$$

Proof ctd.

$$-c - (1-c)\left(-c - \frac{1}{2}c^2 - \frac{1}{3}c^3 - \dots\right)$$

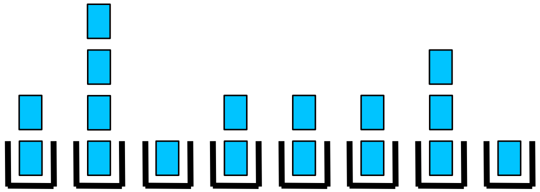
$$-\overset{\checkmark}{c} + \overset{\checkmark}{c} + \frac{1}{2}c^2 + \frac{1}{3}c^3 + \frac{1}{4}c^4 + \frac{1}{5}c^5 \dots$$

$$-c^2 - \frac{1}{2}c^3 - \frac{1}{3}c^4 - \frac{1}{4}c^5 \dots$$

$$= -\frac{1}{2}c^2 - \left(\frac{1}{2} - \frac{1}{3}\right)c^3 - \left(\frac{1}{3} - \frac{1}{4}\right)c^4 - \left(\frac{1}{4} - \frac{1}{5}\right)c^5 \dots$$

$$< -\frac{1}{2}c^2$$

Balls and Bins



Lemma

If m= k n ln n Balls are randomly placed in n bins:

1. Then for all c>k the probability that more than c ln n balls are in a bin is at most $O(n^{-c'})$ for a constant c'>0.
2. Then for all c<k the probability that less than c ln n balls are in a bin is at most $O(n^{-c'})$ for a constant c'>0.

Proof:

Consider a bin and the Bernoulli experiment B(k n ln n,1/n) and expectation: μ = m/n = k ln n

1. Case: c>2k

$$\mathrm{P}[X \geq c \ln n] = \mathrm{P}[X \geq (1+(c/k-1))k \ln n]$$
$$\leq e^{-\frac{1}{3}(c/k-1)k \ln n} \leq n^{-\frac{1}{3}(c-k)}$$

2. Case: k<c<2k

$$\mathrm{P}[X \geq c \ln n] = \mathrm{P}[X \geq (1+(c/k-1))k \ln n]$$
$$\leq e^{-\frac{1}{3}(c/k-1)^2 k \ln n} \leq n^{-\frac{1}{3}(c-k)^2}$$

3. Case: c<k

$$\mathrm{P}[X \leq c \ln n] = \mathrm{P}[X \leq (1-(1-c/k))k \ln n]$$
$$\leq e^{-\frac{1}{2}(1-c/k)^2 k \ln n} \leq n^{-\frac{1}{2}(k-c)^2/k}$$

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